RANKINE-HUGONIOT SHOCK JUMP CONDITIONS

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ABSTRACT

Subject headings: shocks

1. CONSERVATION EQUATIONS

$$j \equiv \rho_1 u_1 = \rho_2 u_2 \tag{1}$$

$$r_1 u_1 (\frac{1}{2}\rho_1 u_1^2 + w_1) = r_2 u_2 (\frac{1}{2}\rho_2 u_2^2 + w_2) \tag{2}$$

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \tag{3}$$

$$w = \gamma P/\rho(\gamma - 1) \tag{4}$$

Let $V \equiv 1/\rho$

$$w = \gamma PV/(\gamma - 1) \tag{5}$$

2. SOME PRELIMINARY RELATIONS

Now begins the algebra with some preliminary steps by taking equation 3) and eliminating all references to ρ by using its inverse V:

$$P_1 + V_1 \rho_1^2 v_1^2 = P_2 + V_2 \rho_2^2 v_2^2 \tag{6}$$

but we can eliminate all v^2, using equation 1) and the definition of j:

$$P_1 + V_1 j^2 = P_2 + V_2 j^2 \tag{7}$$

Solving for j^2 gives:

$$V_1 j^2 - V_2 j^2 = P_2 - P_1 \tag{8}$$

and

$$j^2(V_1 - V_2) = P_2 - P_1 \tag{9}$$

and finally

$$j^2 = (P_2 - P_1)/(V_1 - V_2) \tag{10}$$

A second preliminary relation for the difference in velocity is derived from equation 1) which can be rearranged to give $u_1 = j/r_1 = jV_1$ and similarly $u_2 = j/r_2 = jV_2$:

$$u_1 - u_2 = jV_1 - jV_2 \tag{11}$$

$$u_1 - u_2 = j(V_1 - V_2) \tag{12}$$

But from equation 10) we have the expression for j^2 in terms if P and V, so we can rewrite the velocity difference as:

$$u_1 - u_2 = \left((P_2 - P_1)/(V_1 - V_2)\right)^{1/2}(V_1 - V_2) \tag{13}$$

$$u_1 - u_2 = ((P_2 - P_1)(V_1 - V_2))^{1/2} \tag{14}$$

3. RATIOS V_2/V_1 AND T_2/T_1 AS A FUNCTION OF γ AND P

3.1. V_2/V_1

We begin by deriving the expression for V_1/V_2 starting from equation 2) and using equations 1), 10), and 5):

$$\rho_1 u_1 (\frac{1}{2}\rho_1 u_1^2 + w_1) = \rho_2 u_2 (\frac{1}{2}\rho_2 u_2^2 + w_2) \tag{15}$$

But the leading expressions are both equal to each other by equation 1), i.e., $\rho_1 u_1 = \rho_2 u_2$ leaving the expression:

$$\frac{1}{2}\rho_1 u_1^2 + w_1 = \frac{1}{2}\rho_2 u_2^2 + w_2 \tag{16}$$

Using equation 1) again, we eliminate all v^2 since $\rho u = j$ or equivalantly $u/V = j$ or finally $u^2 = j^2 V^2$:

$$\frac{1}{2} j^2 V_1^2 + w_1 = \frac{1}{2} j^2 V_2^2 + w_2 \tag{17}$$

rearranging the previous equation we have:

$$w_1 - w_2 + \frac{1}{2} j^2 \left(V_1^2 - V_2^2\right) = 0 \tag{18}$$

but from eq. 10, $j^2 = (P_2 - P_1)/(V_1 - V_2)$ and therefore,

$$w_1 - w_2 + \frac{1}{2}\left(V_1^2 - V_2^2\right)\left(\frac{P_2 - P_1}{V_1 - V_2}\right) = 0 \tag{19}$$

$$w_1 - w_2 + \frac{1}{2}(V_1 - V_2)(V_1 + V_2)\left(\frac{P_2 - P_1}{V_1 - V_2}\right) = 0 \tag{20}$$

$$w_1 - w_2 + \frac{1}{2}(V_1 + V_2)(P_2 - P_1) = 0 \tag{21}$$

but we can also eliminate w since $w = \gamma PV/(\gamma - 1)$:

$$\frac{\gamma P_1 V_1}{(\gamma - 1)} - \frac{\gamma P_2 V_2}{(\gamma - 1)} + \frac{1}{2}(V_1 + V_2)(P_2 - P_1) = 0 \tag{22}$$

Expanding all terms and multiplying through by $\gamma - 1$), we find

$$\gamma P_1 V_1 - \gamma P_2 V_2 + \frac{(\gamma - 1)}{2}\left[P_2 V_1 - P_1 V_1 + P_2 V_2 - P_1 V_2\right] = 0 \tag{23}$$

Now combining the two terms with $P_1 V_1$ and the two terms with $P_2 V_2$ yields:

$$\left(\gamma - \frac{(\gamma - 1)}{2}\right) P_1 V_1 + \left(\frac{(\gamma - 1)}{2} - \gamma\right) P_2 V_2 + \left(\frac{(\gamma - 1)}{2}\right) P_2 V_1 - \left(\frac{(\gamma - 1)}{2}\right) P_1 V_2 = 0 \tag{24}$$

Simplifying and multiplying by 2, we find:

$$(\gamma + 1) P_1 V_1 - (\gamma + 1) P_2 V_2 + (\gamma - 1) P_2 V_1 - (\gamma - 1) P_1 V_2 = 0 \tag{25}$$

Now, we wish to solve for V_2/V_1 so we collect terms with the numerator on the left and with the denominator on the right:

$$(\gamma + 1) P_2 V_2 + (\gamma - 1) P_1 V_2 = (\gamma + 1) P_1 V_1 + (\gamma - 1) P_2 V_1 \tag{26}$$

Factoring V_2 on the left and V_1 on the right,

$$V_2\left((\gamma + 1) P_2 + (\gamma - 1) P_1\right) = V_1\left((\gamma + 1) P_1 + (\gamma - 1) P_2\right) \tag{27}$$

Dividing to find V_2/V_1, our next major result, we have:

$$\frac{V_2}{V_1} = \frac{(\gamma + 1) P_1 + (\gamma - 1) P_2}{(\gamma + 1) P_2 + (\gamma - 1) P_1} \tag{28}$$

3.2. T_2/T_1

The next step is to derive the ratio of the temperatures. This follows simply and directly from the ratio of the specific volumes (densities). We use the ideal gas law and equation 28):

Using the ideal gas law we can write the rati of the gas temperatures:

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2} \tag{29}$$

$$\frac{T_2}{T_1} = \frac{P_2V_2}{P_1V_1} \tag{30}$$

and substituting the expression for V_2/V_1 from equation 28) above gives:

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\left(\frac{(\gamma+1)P_1+(\gamma-1)P_2}{(\gamma+1)P_2+(\gamma-1)P_1}\right) \tag{31}$$

4. VELOCITIES

The goal now is to use the above expressions, to derive expressions for the velocities and to introduce into them the Mach number $M \equiv v_1/c_1$.

4.1. *A warm up - another expression for* j^2

We have from before (see equation 10) that j^2 can be written as:

$$j^2 = (P_2-P_1)/(V_1-V_2) \tag{32}$$

but we can now eliminate $V2$ using equation 28):

$$j^2 = \frac{(P_2-P_1)}{V_1-V_1\left(\frac{(\gamma+1)P_1+(\gamma-1)P_2}{(\gamma+1)P_2+(\gamma-1)P_1}\right)} \tag{33}$$

Factoring V_1 in the denominator and clearing the fraction in the denominator gives:

$$j^2 = \frac{(P_2-P_1)\big((\gamma+1)P_2+(\gamma-1)P_1\big)}{V_1\big((\gamma+1)P_2+(\gamma-1)P_1-(\gamma+1)P_1-(\gamma-1)P_2\big)} \tag{34}$$

$$j^2 = \frac{(P_2-P_1)\big((\gamma+1)P_2+(\gamma-1)P_1\big)}{V_1\big(2P_2-2P_1\big)} \tag{35}$$

and finally,

$$j^2 = \frac{(\gamma+1)P_2+(\gamma-1)P_1}{2V_1} \tag{36}$$

4.2. u_1^2

Returning to the mass conservation equation 1), we can derive an expression for u_1^2. We start with equation 1) and the definition of j:

$$j = u_1\rho_1 \quad \text{or equivalently} \quad u_1 = jV_1 \tag{37}$$

Squaring this equation we have:

$$u_1^2 = j^2 V_1^2 \tag{38}$$

Substituting equation 36) for j^2 gives:

$$u_1^2 = \frac{V_1}{2}\left((\gamma+1)P_2 + (\gamma-1)P_1\right) \tag{39}$$

4.3. v_2^2

As with u_1^2, we derive a similar expression for u_2^2 starting from mass conservation, equation 1) and the definition for j:

$$j = u_2\rho_2 \quad \text{or equivalently} \quad u_2 = jV_2 \tag{40}$$

Squaring this equation we have:

$$u_2^2 = j^2 V_2^2 \tag{41}$$

Substituting equation 36) for j^2 and using equation 28) to eliminate V_2 we proceed to find u_2^2. First the expression for j^2 is

$$j^2 = \frac{(\gamma+1)P_2 + (\gamma-1)P_1}{2V_1} \tag{42}$$

and equation 28) gives us for V_2

$$V_2 = V_1 \frac{(\gamma+1)P_1 + (\gamma-1)P_2}{(\gamma+1)P_2 + (\gamma-1)P_1} \tag{43}$$

Substituting for j^2 and V_2 in equation 41) gives:

$$u_2^2 = \frac{(\gamma+1)P_2 + (\gamma-1)P_1}{2V_1} V_1^2 \left(\frac{(\gamma+1)P_1 + (\gamma-1)P_2}{(\gamma+1)P_2 + (\gamma-1)P_1}\right)^2 \tag{44}$$

$$u_2^2 = \frac{V_1}{2}\left(\frac{\left((\gamma+1)P_1 + (\gamma-1)P_2\right)^2}{(\gamma+1)P_2 + (\gamma-1)P_1}\right) \tag{45}$$

5. EXPRESSIONS WITH THE MACH NUMBER $M \equiv U_1/C_1$

The final steps are made by inserting expressions for the sound speed and Mach number. In the undisturbed gas,

$$c_1 = (\gamma P_1/r_1)^{1/2} \tag{46}$$

$$M_1 \equiv u_1/c_1 \tag{47}$$

5.1. P_2/P_1 *as a function of* M

We begin with the expression for U_1^2 from equation 39):

$$u_1^2 = \frac{V_1}{2}\left((\gamma+1)P_2 + (\gamma-1)P_1\right) \tag{48}$$

Factoring out P_2 on the right gives:

$$u_1^2 = \frac{V_1 P_1}{2}\left((\gamma+1)P_2/P_1 + (\gamma-1)\right) \tag{49}$$

Then, we use the definition of the Mach number and sound speed to give an espression for P_1V_1 as follows:

$$M_1^2 = \frac{u_1^2}{c_1^2} = \frac{u_1^2}{\gamma P_1/\rho_1} = \frac{u_1^2}{\gamma P_1 V_1} \tag{50}$$

Solving for P_1V_1, we have:

$$P_1 V_1 = \frac{u_1^2}{\gamma M_1^2} \tag{51}$$

Finally substituting this result into equation 49),

$$u_1^2 = \frac{u_1^2}{2\gamma M_1^2}\left((\gamma+1)P_2/P_1 + (\gamma-1)\right) \tag{52}$$

$$2\gamma M_1^2 = (\gamma+1)P_2/P_1 + (\gamma-1) \tag{53}$$

$$M_1^2 = \frac{(\gamma+1)}{2\gamma}\frac{P_2}{P_1} + \frac{(\gamma-1)}{2\gamma} \tag{54}$$

or solving for P_2/P_1:

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{(\gamma+1)} \tag{55}$$

5.2. $\rho_2/\rho_1 = v_1/v_2$ *as a function of* M

Next we derive the ratio of the gas densities using the expressions for u_1^2 and u_1^2 (equations 39 and 45) We start with the mass conservation equation which yields:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \sqrt{\frac{u_1^2}{u_2^2}} \tag{56}$$

Then we substitute for the squares of the velocities from equations 39 and 45:

$$\frac{\rho_2}{\rho_1} = \left(\frac{\frac{V_1}{2}\left((\gamma+1)P_2+(\gamma-1)P_1\right)}{\frac{V_1}{2}\left(\frac{\left((\gamma+1)P_1+(\gamma-1)P_2\right)^2}{(\gamma+1)P_2+(\gamma-1)P_1}\right)} \right)^{\frac{1}{2}} \tag{57}$$

which miraculously reduces to the simple expression:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma-1)P_1+(\gamma+1)P_2}{(\gamma+1)P_1+(\gamma-1)P_2} \tag{58}$$

We wish to rewrite this expression in terms of the ratio of the pressures. Factoring P_1 from both the numerator and denominator yeilds the desired result:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma-1)+(\gamma+1)P_2/P_1}{(\gamma+1)+(\gamma-1)P_2/P_1} \tag{59}$$

But we have already derived the expression for P_1/P_2 in terms of M_1 (equation 55) which is:

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)} \tag{60}$$

When this equation is used to eliminate P_1/P_2 from equation 59), we find:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma-1)+2\gamma M_1^2-(\gamma-1)}{(\gamma+1)+(\gamma-1)\left(\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}\right)} \tag{61}$$

Simplifying

$$\frac{\rho_2}{\rho_1} = \frac{2\gamma M_1^2(\gamma+1)}{(\gamma+1)^2+(\gamma-1)\left(2\gamma M_1^2-(\gamma-1)\right)} \tag{62}$$

$$\frac{\rho_2}{\rho_1} = \frac{2\gamma M_1^2(\gamma+1)}{(\gamma+1)^2-(\gamma-1)^2+(\gamma-1)\left(2\gamma M_1^2\right)} \tag{63}$$

$$\frac{\rho_2}{\rho_1} = \frac{2\gamma M_1^2(\gamma+1)}{(\gamma+1-\gamma+1)(\gamma+1+\gamma+1)+(\gamma-1)\left(2\gamma M_1^2\right)} \tag{64}$$

$$\frac{\rho_2}{\rho_1} = \frac{2\gamma M_1^2(\gamma+1)}{4\gamma+(\gamma-1)\left(2\gamma M_1^2\right)} \tag{65}$$

Finally,

$$\frac{\rho_2}{\rho_1} = \frac{M_1^2(\gamma+1)}{2+(\gamma-1)M_1^2} \tag{66}$$

5.3. T_2/T_1 *as a function of M*

The last step is to derive the expression for T_2/T_1. We begin with the ideal gas law:

$$P_2 \propto \rho_2 T_2 \quad \text{and} \quad P_1 \propto \rho_1 T_1 \tag{67}$$

Solving each of the expressions for the temperature and dividing we have:

$$\frac{T_2}{T_1} = \frac{P_2\rho_1}{P_1\rho_2} = \frac{P_2}{P_1} \Big/ \frac{\rho_2}{\rho_1} \tag{68}$$

However, we already have expressions or P_2/P_1 and ρ_2/ρ_1 (equations 55 and 66) which yields:

$$\frac{T_2}{T_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{(\gamma+1)} \Big/ \frac{M_1^2(\gamma+1)}{2+(\gamma-1)M_1^2} \tag{69}$$

$$\frac{T_2}{T_1} = \frac{\left(2\gamma M_1^2 - (\gamma-1)\right)\left(2+(\gamma-1)M_1^2\right)}{(\gamma+1)M_1^2(\gamma+1)} \tag{70}$$

Finally, the last desired expression is:

$$\frac{T_2}{T_1} = \frac{\left(2\gamma M_1^2 - (\gamma-1)\right)\left((\gamma-1)M_1^2+2\right)}{(\gamma+1)^2 M_1^2} \tag{71}$$