Ages of stellar populations from color-magnitude diagrams

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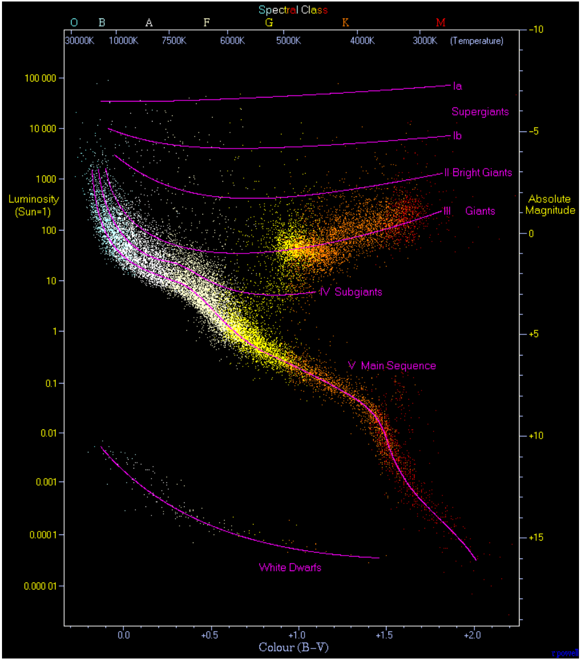
Welcome!

Today we will look at using hierarchical Bayesian modeling to make inference about the properties of stars; most notably the age and mass of groups of stars. Complete with a brief dummies (statisticians) guide to the Astronomy behind it.


Spectral Class
O B A F G K M
30000K 10000K 7500K 6000K 5000K 4000K 3000K (Temperature)
Luminosity (Sun=1)
100 000
10 000
1000
100
10
1
0.1
0.01
0.001
0.0001
0.000 01
Absolute Magnitude
−10
−5
0
+5
+10
+15
Ia
Supergiants
Ib
II Bright Giants
III Giants
IV Subgiants
V Main Sequence
White Dwarfs
0.0
+0.5
+1.0
+1.5
+2.0
Colour (B−V)
cpowell

Isochrones For Dummies/Statisticians

Given the mass, age and metallicity of a star, we 'know' what its 'ideal' observation should be i.e., where it should be on the CMD.

The tables of these 'ideal' observations are called *isochrone tables*.

Why are they only 'ideal' colours/magnitudes?

Observational Error

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1. These are relatively well understood – and can be considered to be Gaussian with *known* standard deviation.
2. Importantly, we can characterize the standard deviation as a function of the observed data.
   i.e., given $\mathbf{Y}_i = (Y_{iB}, Y_{iV}, Y_{iI})^T$ we have $\sigma_i = \sigma(\mathbf{Y}_i)$.

The Observed Data

We observe (depending on the experiment) p different colours/magnitudes for n stars.

Although it is equally straightforward to model colours $U - B, B - V$ etc., and magnitudes B, V, etc., we will stick with *magnitudes*.

The (known) standard deviations in each band are also recorded for each observation.

We also observe that we observe the n stars in the dataset and that we didn't observe any others!

The Likelihood I

$$y_i = \begin{pmatrix} \frac{1}{\sigma_i^{(B)}} B_i \\ \frac{1}{\sigma_i^{(V)}} V_i \\ \frac{1}{\sigma_i^{(I)}} I_i \end{pmatrix} \Bigg| A_i, M_i, Z \sim N\left(\tilde{f}_i, \mathbf{R}\right) \qquad i = 1, \ldots, n \qquad (1)$$

Where,

$$\tilde{f}_i = \begin{pmatrix} \frac{1}{\sigma_{Bi}} \cdot f_b(A_i, M_i, Z) \\ \frac{1}{\sigma_{Vi}} \cdot f_v(A_i, M_i, Z) \\ \frac{1}{\sigma_{Ii}} \cdot f_i(A_i, M_i, Z) \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} 1 & \rho^{(BV)} & \rho^{(BI)} \\ \rho^{(BV)} & 1 & \rho^{(VI)} \\ \rho^{(BI)} & \rho^{(VI)} & 1 \end{pmatrix}.$$

The Likelihood II

Let $S_i = 1$ if star i is observed, $S_i = 0$ otherwise.

$$S_i | \mathbf{Y}_i \sim \mathrm{Bernoulli}\left(p\left(\mathbf{Y}_i\right)\right) \tag{2}$$

where $p(\mathbf{Y}_i)$ is the probability of a star of a given magnitude being unobserved (provided by Astronomers).

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Note: We can also have $S_i = (S_{iB}, S_{iV}, S_{iI})^T$ and allow for some stars to be observed only in a subset of the bands.

The Parameters

Mass

Before we have any data, the prior distributions of mass and age are still not independent. We know *a priori* that old stars cannot have large mass, likewise for very young stars. Hence, we specify the prior on mass conditional on age:

$$p\left(M_i | A_i, M_{min}, M_{max}\left(A_i\right), \alpha\right) \propto \frac{1}{M_i^{\alpha}} \cdot \mathbf{1}_{\{M_i \in [M_{min}, M_{max}(A_i)]\}} \quad (3)$$

i.e. $M_i | A_i, M_{min}, M_{max}\left(A_i\right), \alpha \sim \text{Truncated-Pareto}.$

Age

For age we assume the following hierarchical structure:

$$A_i|\mu_A, \sigma_A^2 \overset{iid}{\sim} N\left(\mu_A, \sigma_A^2\right) \tag{4}$$

where $A_i = \log_{10}(\mathrm{Age})$, with μ_A and σ_A^2 hyperparameters...

Metallicity

Denoted by Z_i.

Assumed to be known and common to all stars i.e., $Z_i = Z = 4$

Hyperparameters

Next, we model the hyperparameters with the simple conjugate form:

$$\mu_A|\sigma_A^2 \sim N\left(\mu_0, \frac{\sigma_A^2}{\kappa_0}\right), \qquad \sigma_A^2 \sim Inv-\chi^2\left(\nu_0, \sigma_0^2\right) \qquad (5)$$

Where μ_0, κ_0, ν_0 and σ_0^2 are fixed by the user to represent prior knowledge (or lack of).

Correlation

We assume a uniform prior over the space of positive definite correlation matrices.

This isn't quite uniform on each of $\rho^{(BV)}, \rho^{(BI)}$ and $\rho^{(VI)}$, but it is very close.

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- This censoring can bias conclusions about the stellar cluster parameters.
- Since magnitudes are functions of photon arrivals, the censoring is stochastic.

Putting it all together

$$S_{ij}|\mathbf{Y}_i \sim \text{Bernoulli}\left(p\left(\mathbf{Y}_i\right)\right) \qquad i = 1, \ldots, n, n+1, \ldots, n+n_{mis} \quad j \in \{B, V, I\}$$

$$y_i = \left(\begin{array}{c} \frac{1}{\sigma_i^{(B)}} B_i \\ \frac{1}{\sigma_i^{(V)}} V_i \\ \frac{1}{\sigma_i^{(I)}} I_i \end{array} \right) \Bigg| A_i, M_i, Z \sim N\left(\tilde{f}_i, \mathbf{R}\right) \qquad i = 1, \ldots, n, n+1, \ldots, n+n_{mis}$$

$$M_i|A_i, M_{min}, \alpha \sim \text{Truncated-Pareto}\left(\alpha - 1, M_{min}, M_{max}\left(A_i\right)\right)$$

$$A_i|\mu_A, \sigma_A^2 \overset{iid}{\sim} N\left(\mu_A, \sigma_A^2\right)$$

$$\mu_A|\sigma_A^2 \sim N\left(\mu_0, \frac{\sigma_A^2}{\kappa_0}\right), \qquad \sigma_A^2 \sim Inv-\chi^2\left(\nu_0, \sigma_0^2\right)$$

$$p(\mathbf{R}) \propto \mathbf{1}_{\{\mathbf{R}\, p.d.\}}$$

Observed-Data Posterior

The product of the densities on the previous slide gives us the *complete-data posterior*. Alas, we don't observe all the stars, and n_{mis} is an unknown parameter. For now, lets just condition on n_{mis}. We have:

$$\begin{aligned}
\mathbf{W}_{obs} &= \left\{ n, \mathbf{y}_{[1:n]} = (\mathbf{y}_1, \ldots, \mathbf{y}_n), \mathbf{S} = \{1, \ldots, 1, 0, \ldots, 0)\} \right\} \\
\mathbf{W}_{mis} &= \left\{ m, \mathbf{Y}_{[(n+1):(n+m)]}, \mathbf{M}_{[(n+1):(n+m)]}, \mathbf{A}_{[(n+1):(n+m)]} \right\} \\
\Theta &= \left\{ \mathbf{M}_{[1:n]}, \mathbf{A}_{[1:n]}, \mu_A, \sigma_A^2, \mathbf{R} \right\}
\end{aligned}$$

where $\mathbf{X}_{a:b}$ denotes the vector $(X_a, X_{a+1}, \ldots, X_b)$

Observed-Data Posterior

We want $p(\Theta|\mathbf{W}_{obs})$ but so far we have $p(\Theta, \mathbf{W}_{mis}|\mathbf{W}_{obs})$. So, we integrate out the missing data:

$$p(\Theta|\mathbf{W}_{obs}) = \int p(\Theta, \mathbf{W}_{mis}|\mathbf{W}_{obs})\, d\mathbf{W}_{mis} \tag{6}$$

In practice, this integration is done by sampling from $p(\Theta, \mathbf{W}_{mis}|\mathbf{W}_{obs})$ and retaining only the samples of Θ.

Observed-Data Posterior

We form a Gibbs sampler to sample from $p\left(\Theta, \mathbf{W}_{mis} | \mathbf{W}_{obs}\right)$. Given a current state of our Markov Chain, $\Theta = \Theta^{(t)}$ and $\mathbf{W}_{mis} = \mathbf{W}_{mis}^{(t)}$:

1. Draw $\Theta^{(t+1)}$ from $p\left(\Theta | \mathbf{W}_{mis}^{(t)}, \mathbf{W}_{obs}\right)$ (as before)
2. Draw $\mathbf{W}_{mis}^{(t+1)}$ from $p\left(\mathbf{W}_{mis} | \Theta^{(t+1)}, \mathbf{W}_{obs}\right)$ (new)

SAMPLING $\mathbf{W}_{mis}$

At each iteration of the Gibbs sampler we need to draw the missing data from the appropriate distribution.

In other words, given a bunch of masses, ages, and metallicities of n_{mis} missing stars, find a bunch of $\mathbf{Y}_i$'s that are consistent with that:

$$p_i\left(Y_i|\mathbf{Y}_{[1:n]},\mathbf{M},\mathbf{A},\mu_A,\sigma_A^2\right)\propto[1-\pi\left(\mathbf{Y}_i\right)]\cdot \tag{7}$$

$$\exp\left\{-\frac{1}{2}\left(\mathbf{Y}_i-\tilde{f}(\mathbf{Y}_i;M_i,A_i,Z)\right)^T R^{-1}\left(\mathbf{Y}_i-\tilde{f}(\mathbf{Y}_i;M_i,A_i,Z)\right)\right\} \tag{8}$$

for $i=n+1,\ldots,n+m$.

SAMPLING $\mathbf{W}_{mis}$

Once we have sampled a new set of $\mathbf{Y}_{mis}$, we need to sample the standard deviation of the Gaussian error for those stars.

Here we assume this is a deterministic mapping: $\sigma = \sigma(\mathbf{Y}_i)$.

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6. High dimensional multi-modal, so we also use parallel tempering.

Parallel Tempering

A brief overview of parallel tempering:

The parallel tempering framework involves sampling N chains, with the i^{th} chain of the form:

$$p_i(\theta) = p(\theta|\mathbf{y})^{1/t_i} \propto \exp\left\{-\frac{H(\theta)}{t_i}\right\} \tag{9}$$

As t_i increases the target distributions become flatter.

Simulation Results

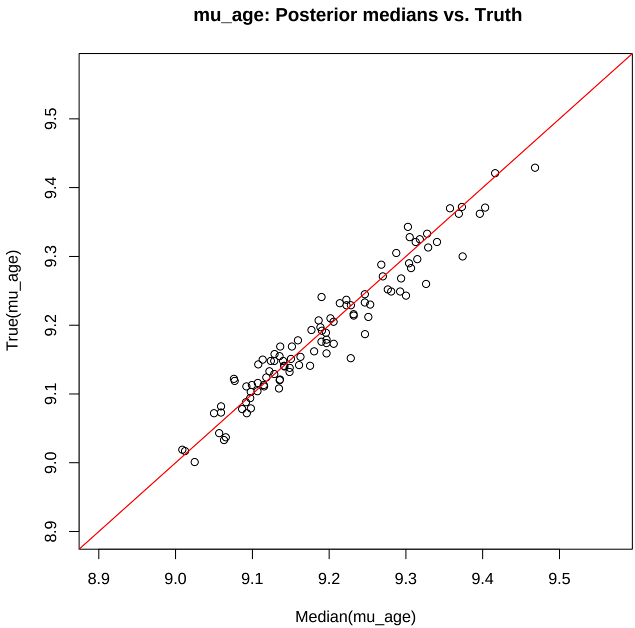
We simulate 100 datasets from the model with $n = 100$:

$$\mu_A = 9.2 \ \sigma_A^2 = 0.01^2$$

$$M_{(min)} = 0.8 \ \alpha = 2.5$$

$$\mathbf{R} = \mathbf{I}$$

$$(\sigma_{B_i}, \sigma_{V_i}, \sigma_{I_i}) \in (0.03, 0.12)$$

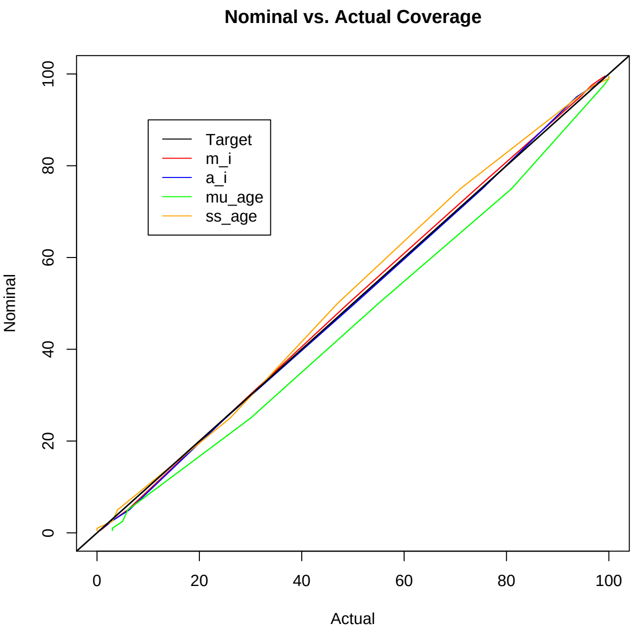

mu_age: Posterior medians vs. Truth
True(mu_age)
Median(mu_age)
8.9
9.0
9.1
9.2
9.3
9.4
9.5

Does it work?

Post_p	0.5	1.0	2.5	5.0	25.0	50.0
m_cover	0.6	1.2	2.8	6.0	25.0	49.1
a_cover	0.4	1.1	2.8	6.3	25.1	50.4
mu_age	3.0	3.0	5.0	6.0	30.0	55.0
ss_age	0.0	0.0	3.0	4.0	26.0	47.0

Does it work?

Post_p	50.0	75.0	95.0	97.5	99.0	99.5
m_cover	49.1	74.1	94.4	96.6	98.5	99.2
a_cover	50.4	75.3	93.7	97.2	99.0	99.3
mu_age	55.0	81.0	97.0	99.0	100.0	100.0
ss_age	47.0	71.0	94.0	97.0	100.0	100.0


Nominal vs. Actual Coverage
Target
m_i
a_i
mu_age
ss_age
Nominal
Actual
0
20
40
60
80
100

Future Work

Some important things still need to be built into the model before it is fit for purpose:

- **Extinction/Absorption:** Shift in observed data
- **Multi-Cluster Models:** Allow for multiple stellar clusters