

Power-Law Estimation via MPS

Theory and Applications

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- 1 The Problem: Unspecified Power-Law Regions in Solar Flare Distributions
- 2 Some Solutions: Maximum Product of Spacings Method
- 3 The Results: MPS Fits to GOES

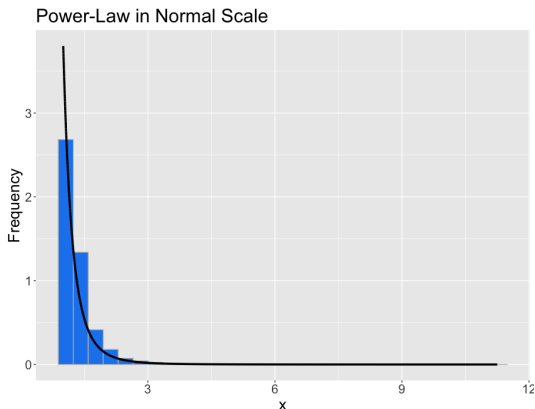
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Power-Law Distributions in Solar Flares

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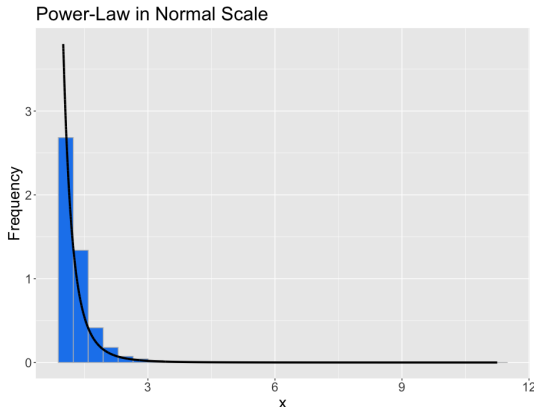
- The functional form of a power-law is given by $f(x) = cx^{-\alpha}$.



Power-Law Distributions in Solar Flares

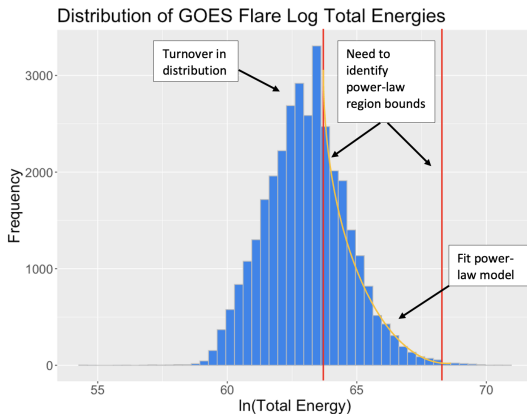
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- In log scale, where $y = \log(x)$, it has the form $g(y) = ke^{-y(\alpha - 1)}$.



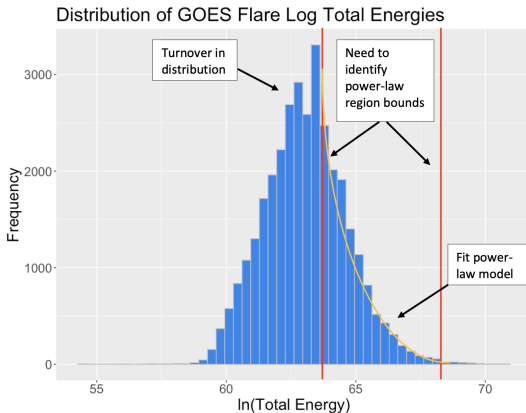
Turnover in the Distribution

- A power-law is only observed over a bounded range to the right of the mode.



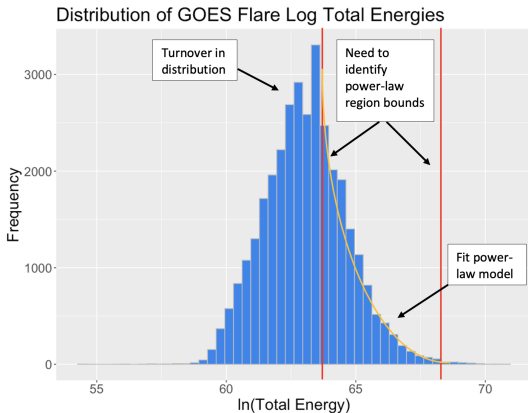
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Turnover in the Distribution

- A power-law is only observed over a bounded range to the right of the mode.
- The turnover observed at lower values is the result of missing data caused by detection sensitivity.
- The start of the power-law is typically visually determined, or by using the K-S statistic (see Clauset et al. 2009).



The Problem

How do we accurately determine the power-law region and estimate ?

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Maximum Product of Spacings Method

- For $X_1 < X_2 < \dots < X_n$, fit a CDF that maximizes the product of spacings: $\prod_{i=0}^n f(F(X_{i+1}) - F(X_i))g$.

Maximum Product of Spacings Method

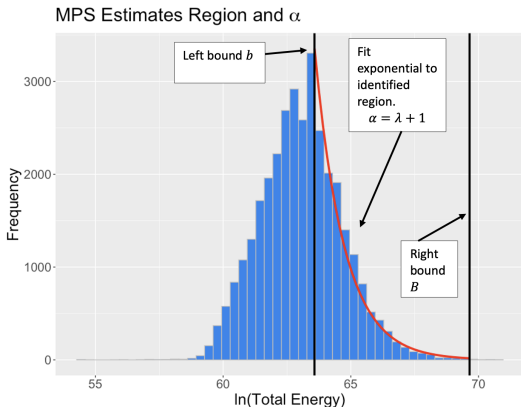
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- We fit a parametric MPS model, and add a penalty term.

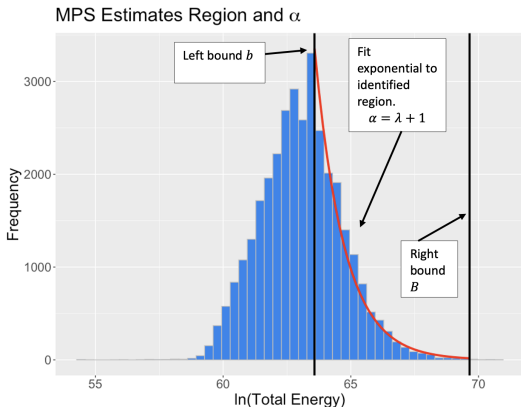
Maximum Product of Spacings Method

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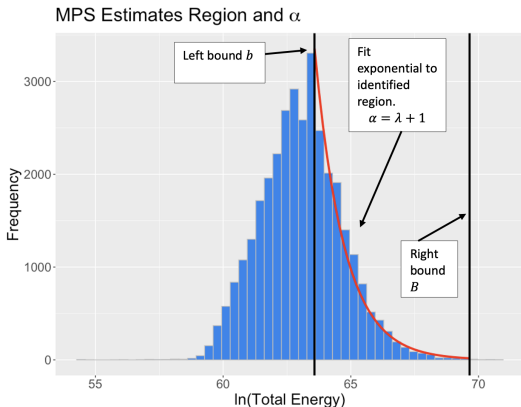
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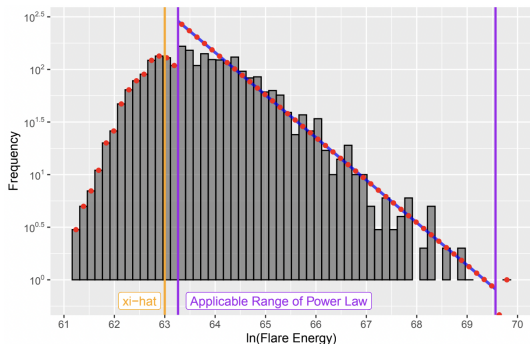
Maximum Product of Spacings Method

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- The Maximum Product of Spacings method estimates these bounds and the rate parameter λ . ($\alpha = \lambda + 1$)



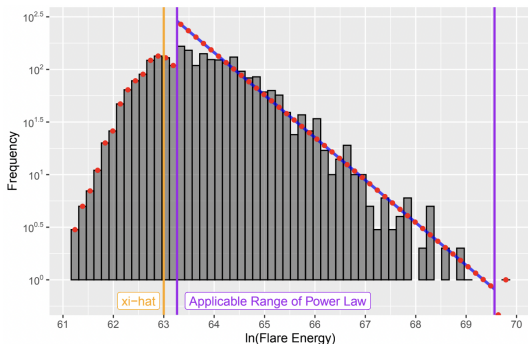
Alternative: Hybrid Parametric Multinomial Model

- Given k pre-specified energy bins, fit a multinomial model using bin probabilities.



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- Given k pre-specified energy bins, fit a multinomial model using bin probabilities.
- Bin probabilities within the power-law region are parameterized by the power-law, outside are not.
- Maximize the likelihood function, estimate $\hat{\xi}$ and lower bound ξ_{\min} .

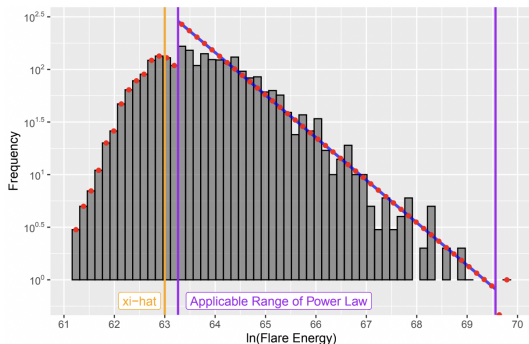
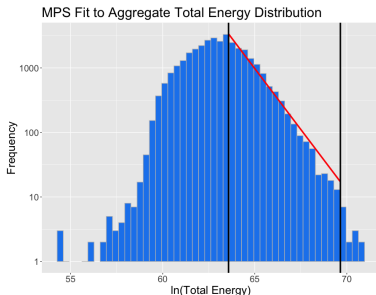


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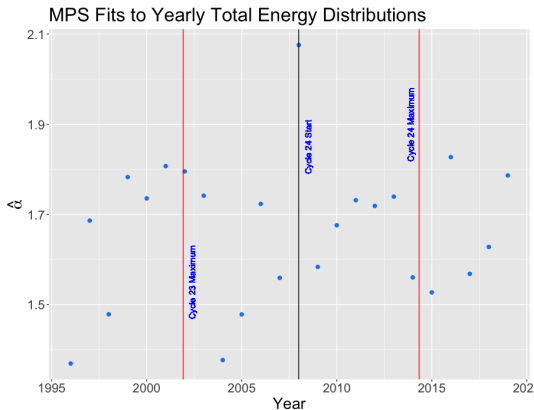
Total Energy - Aggregate and By-Cycle

Data	$\hat{\alpha}$	\hat{b}	\hat{B}	n
Aggregate	1.87	63.58	69.65	33445
Cycle 23	1.85	63.58	69.09	18457
Cycle 24	1.64	62.61	67.10	14988



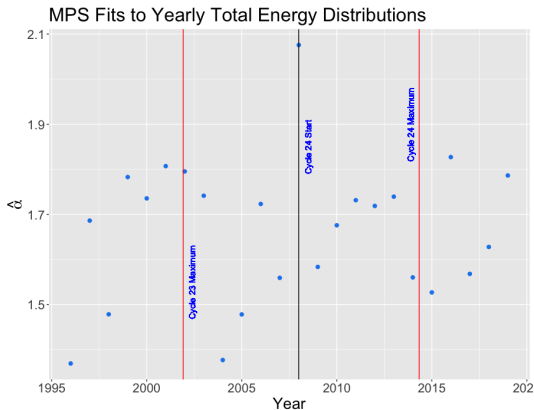
Total Energy - Yearly

- $\hat{\alpha}$ varies between 1.37 and 1.87 throughout the two solar cycles (outlier of $\hat{\alpha} = 2.07$ in 2008, where $n = 88$)



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- $\hat{\alpha}$ varies between 1.37 and 1.87 throughout the two solar cycles (outlier of $\hat{\alpha} = 2.07$ in 2008, where $n = 88$)
- No significant trend in $\hat{\alpha}$ within a solar cycle.



Summary

- We have further developed our model based on the MPS method to estimate the lower and upper bounds of the power-law region, and .

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- For aggregate data, $\hat{\alpha} = 1.87$. Similar to previous estimates close to 1.8.
- For yearly data, $\hat{\alpha}$ varies between 1.37 and 1.87. We identify no clear trend.

Moving Forward

- Further study performance of MPS through simulations.

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- Further study performance of MPS through simulations.
- Fit our power-law models to solar flare property distributions (total energy, peak flux, waiting time).
- Compare power-law models and verify results.



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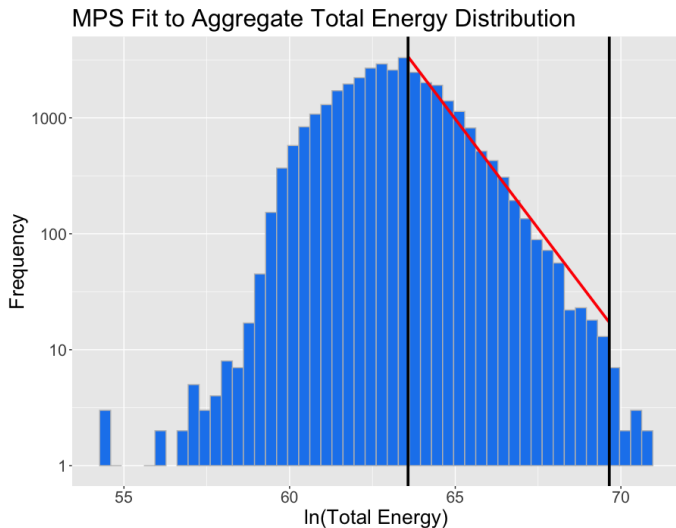


A. Clauset, C. R. Shalizi, and M. E. J. Newman

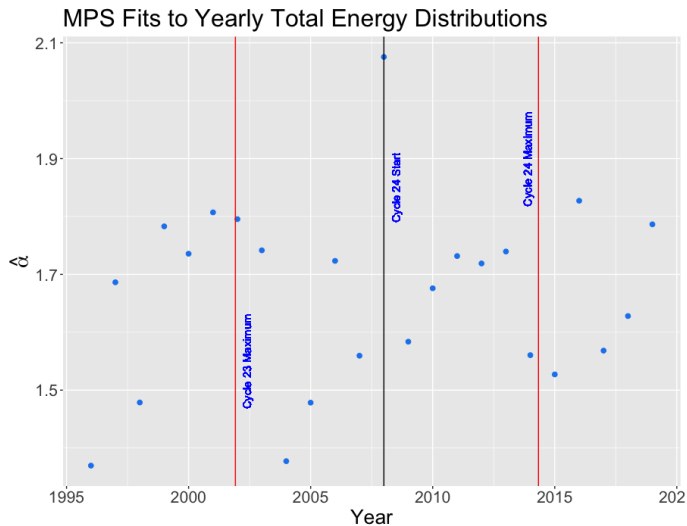
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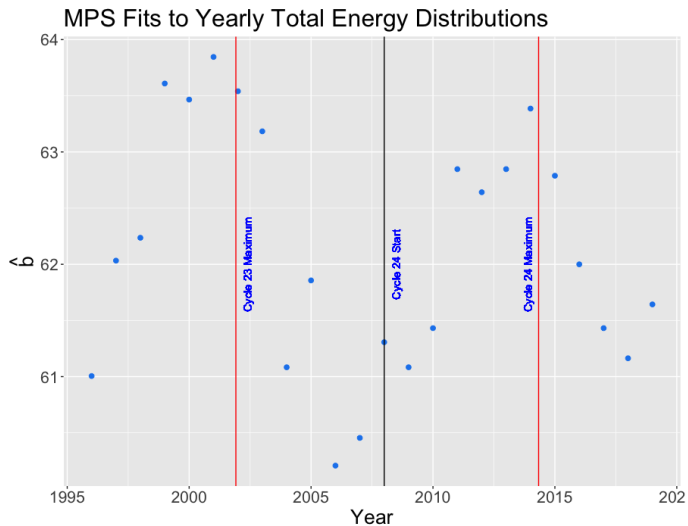
MPS Fit to Aggregate Total Energy



MPS α Fits to Yearly Total Energy

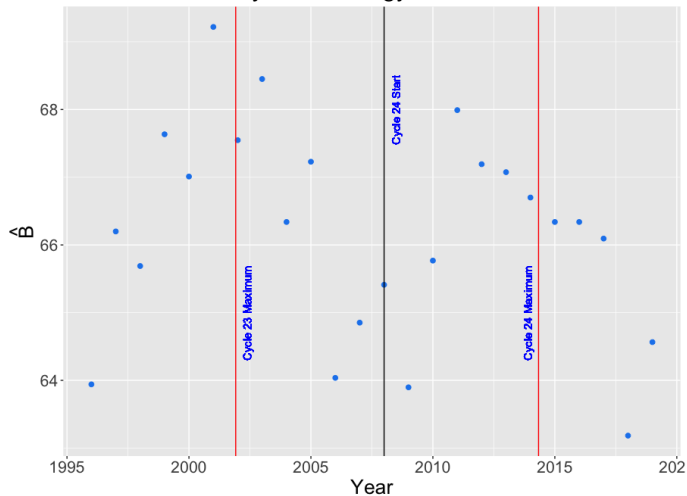


MPS Left Bound Fits to Yearly Total Energy

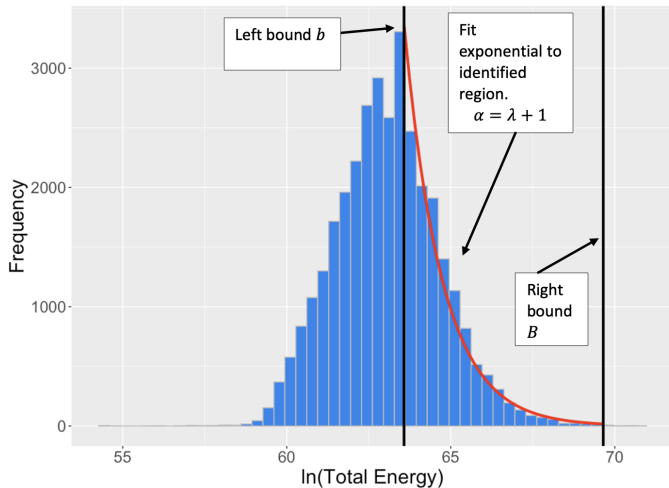


MPS Right Bound Fits to Yearly Total Energy

MPS Fits to Yearly Total Energy Distributions



MPS Estimates Region and α



Semi-Parametric Multinomial Model

