Statistics in X-ray Polarimetry

Overview of some statistics in use or development
for X-ray Polarimetry

Herman L. Marshall (MIT)

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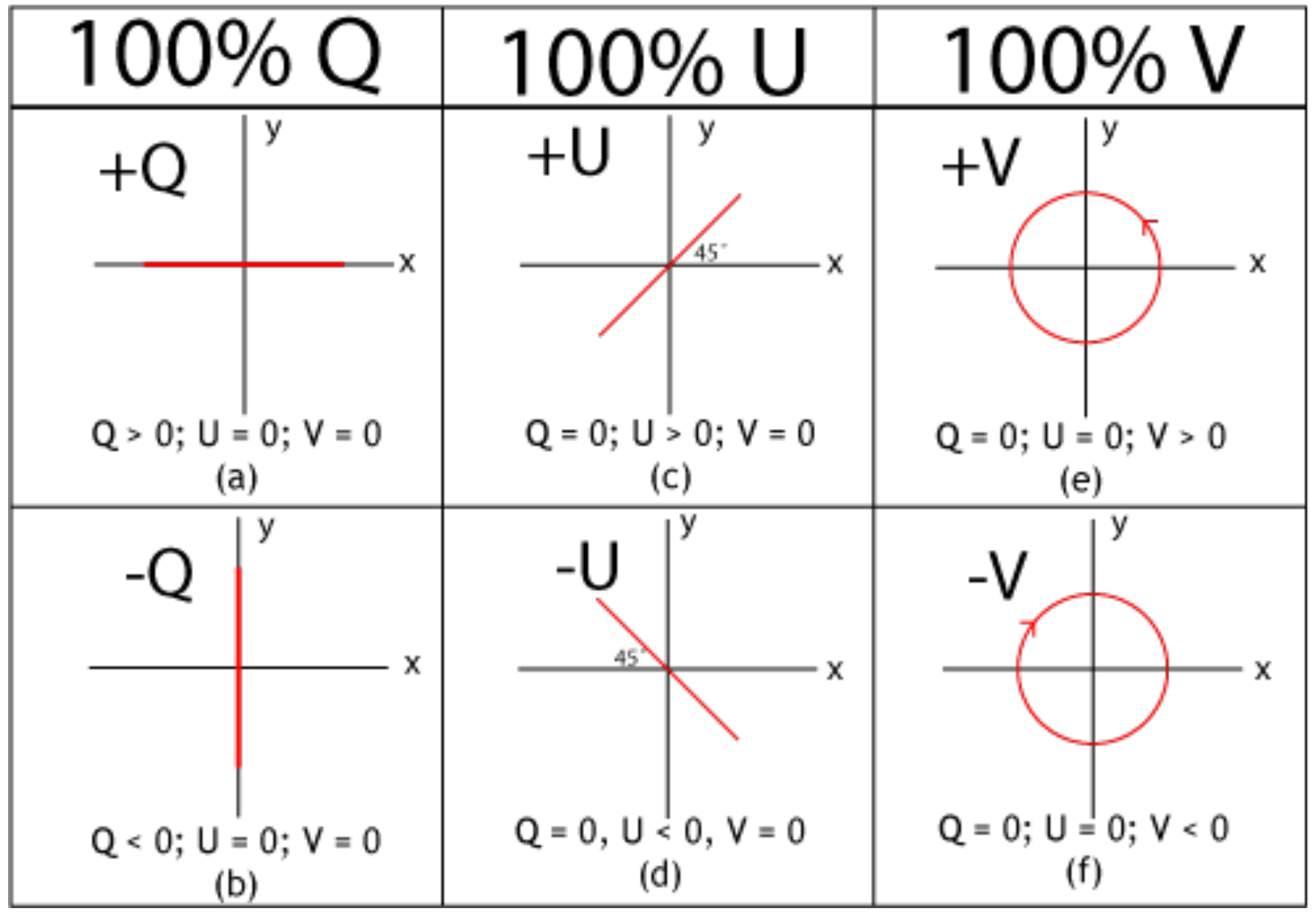
A Few Basics

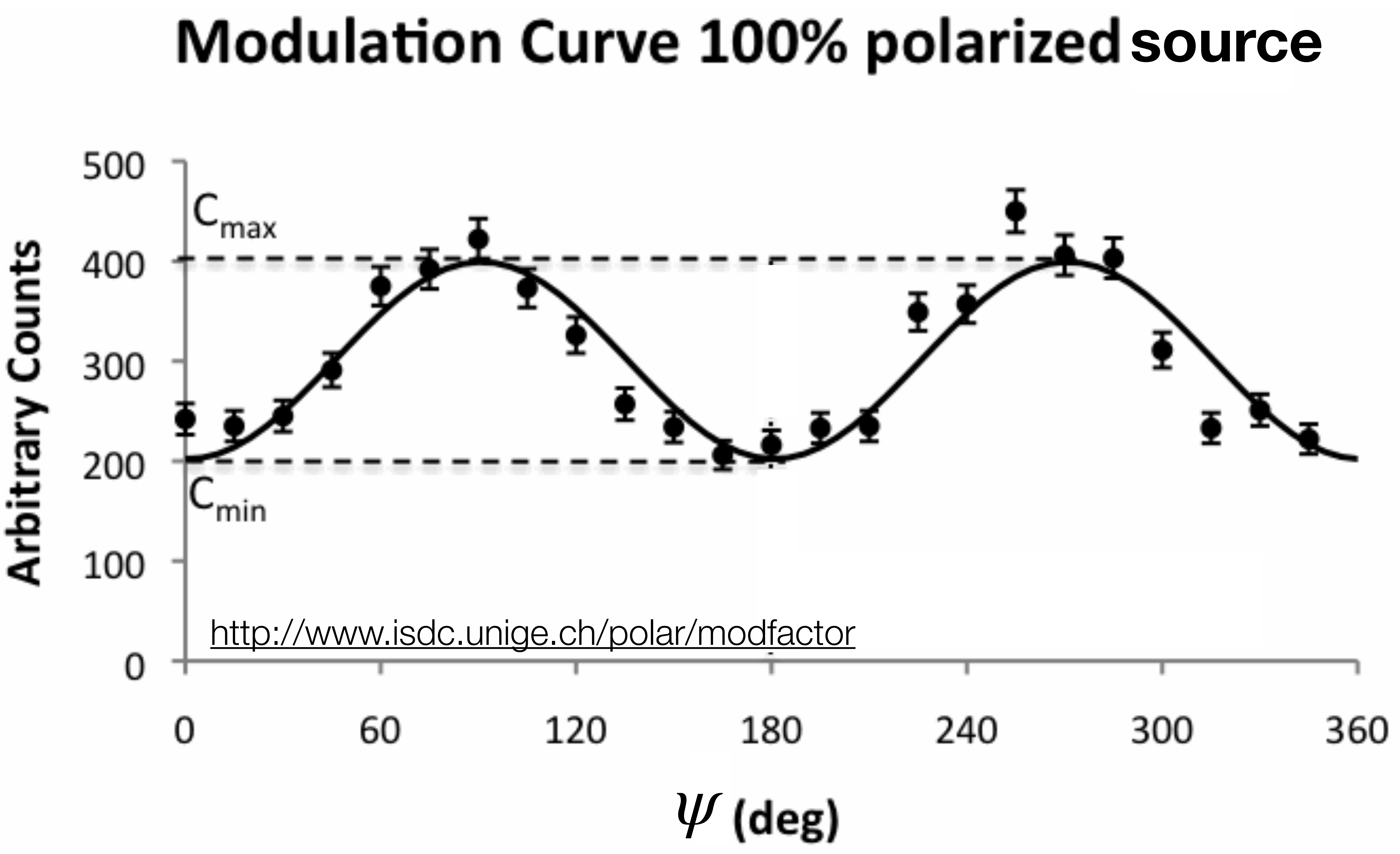
- Stokes parameters are handy:
  - I = total intensity
  - Q, U are orthogonal linearly polarized parts
  - V is circular (+ or -) polarized intensity

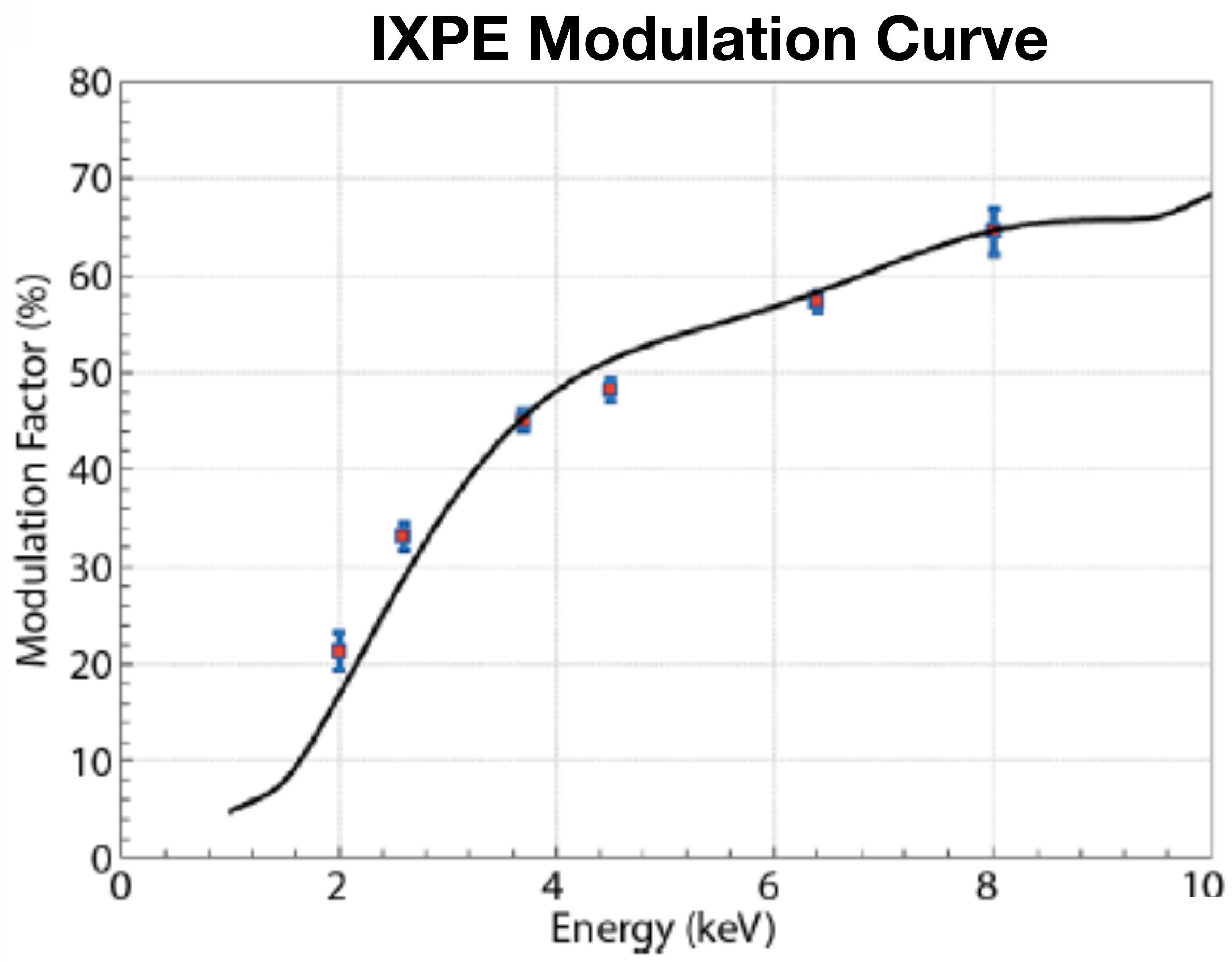


- Common alternative: Π, ϕ
  - $\Pi = (Q^2 + U^2)^{1/2} / I$
  - $\phi = \tan^{-1}(U/Q) = 2 \times$ EVPA
- A beam is "unpolarized" if the photon **set** is randomly polarized ($\Pi = V = 0$)
- MDP = 'Minimum Detectable Polarization' (at 99% conf.) = $\dfrac{4.292\sqrt{N_S + N_B}}{\mu N_S}$

$$4.292 = 2(-\log[0.01])^{1/2}$$

Modulation of Polarized Signals





Modulation Factor = $\mu = (C_{max} - C_{min})/(C_{max} + C_{min})$

$$n(\psi) = \frac{1}{2\pi}(1 + \Pi\mu\cos[2\psi - 2\varphi])$$

Relevant Work

- Elsner, O'Dell, & Weisskopf (2012): Gaussian statistics, BG
- Kislat+ (2015): Unbinned analysis, event weighting
- Strohmayer (2017): Fitting IQU spectra in xspec, mRMF $= \mu R(E; E')$
- Burgess+ (2019): Likelihood method for GRB polarimetry
- Peirson+ (2021): Machine learning to get better μ
- Marshall (2021, 2022): Likelihood method, modeling μ
- Di Marco+ (2022): Event weights using IXPE track ellipticities α
- Gonzales-Caniulef+ (2022): Likelihood method for pulsars
- Marshall (in prep.): Likelihood with BG, nonuniform ψ, mRMF(E, α; E')

Likelihood Formulation (Marshall 2021)

- Expected counts in $dEd\psi$ in time *T* for $Q = qI, U = uI$:
  - $\lambda(E, \psi; f_0, q, u)dEd\psi = [1 + \mu_E(q\cos 2\psi + u\sin 2\psi)]f_E A_E T dEd\psi$, where
    - A_E is the instrument effective area (independent of q or u by definition)
    - $f_E = f_0\eta(E)$ has units of ph/cm^2/s/keV per unit (measured) phase angle, ψ
  - Require $\Pi^2 \equiv q^2 + u^2 \leq 1$ physically (Π is fractional linear polarization)
  - Define $\phi_0 = \tan^{-1}(u/q) = 2\varphi$
- Likelihood: $S(f_0, q, u) = -2\ln\mathscr{L} = -2\sum_i \ln\lambda(E_i, \psi_i) + 2T\int f_E A_E dE \int_0^{2\pi} [1 + \mu(E)(q\cos 2\psi + u\sin 2\psi)]d\psi$

  or $\tilde{S}(q, u) = -2\sum_i \ln(1 + q\mu_i\cos 2\psi_i + u\mu_i\sin 2\psi_i)$
- MDP$_{99}$ $= 4.29/\sqrt{\sum \mu_E^2 C(E)}$ for small $\Pi\mu$

Adding BG to Likelihood (Marshall, in prep)

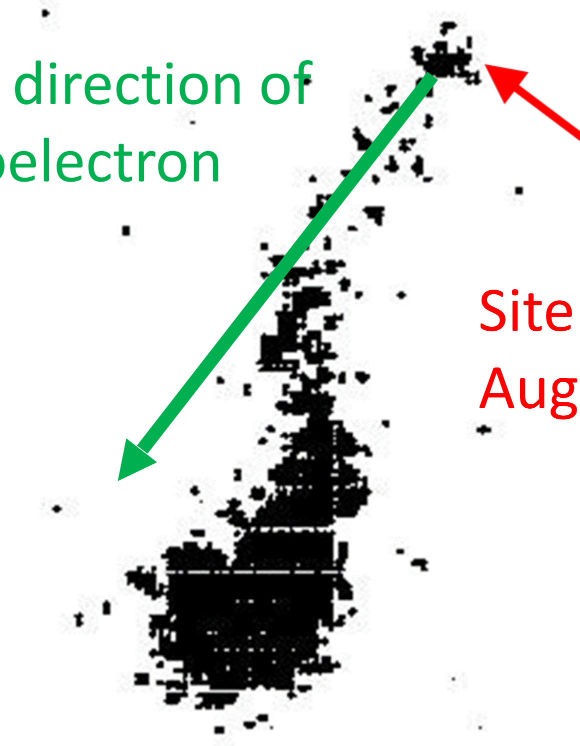
- $\lambda_S(\psi) = \frac{1}{2\pi}\{N_0[1+\mu(q\cos 2\psi + u\sin 2\psi)] + \zeta B\}$ for the $N = C_S + C_B$ counts in the source region, $\lambda_B(\psi) = \frac{B}{2\pi}$ for the N_B counts in the BG region, and $C_B = \zeta B$ is the expected BG in the source region

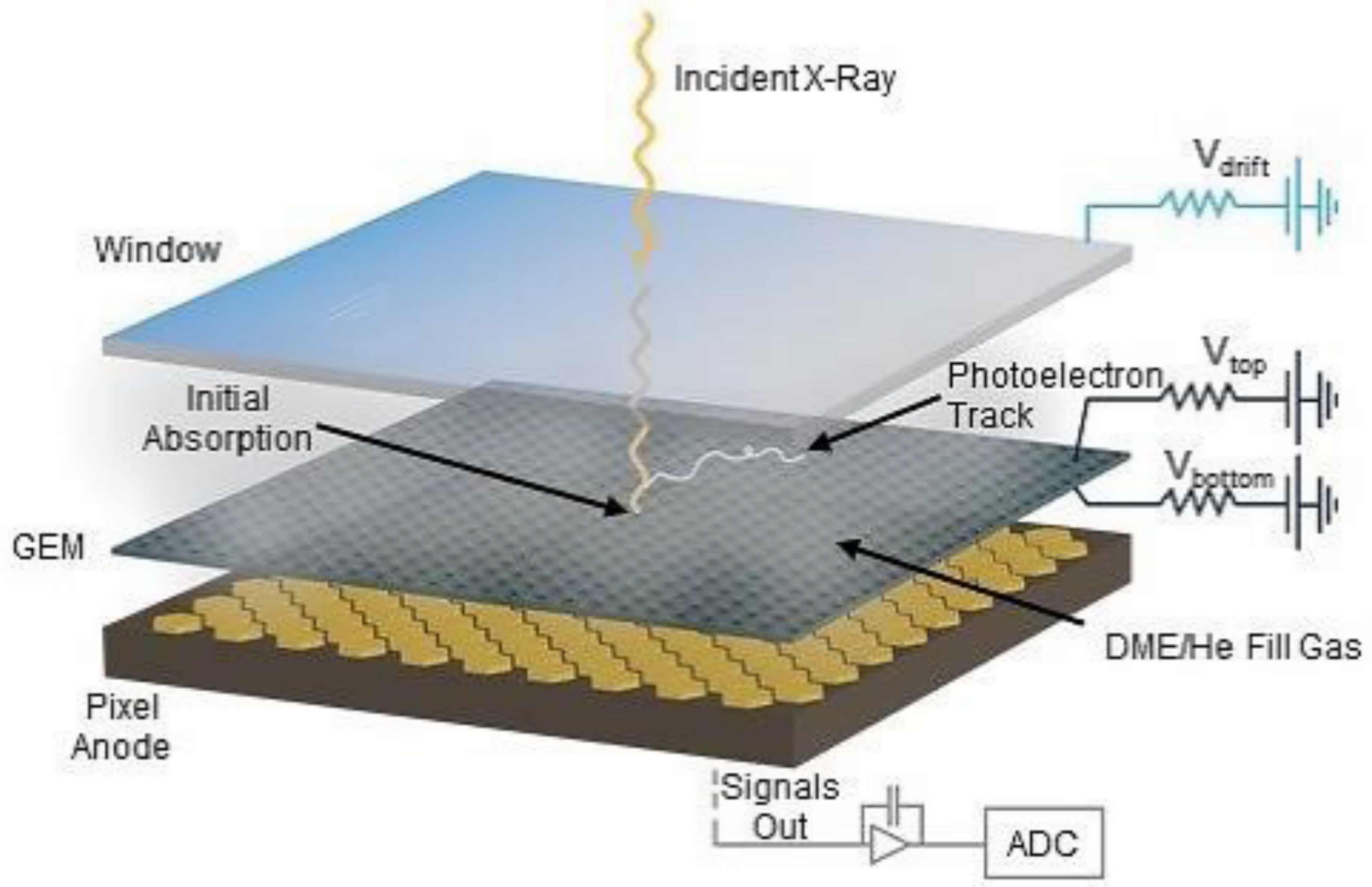
- Likelihood: $S(N_0, q, u) = -2\sum_{i=1}^{N} \ln[N_0(1+qc_i+us_i)+\zeta B] + 2N_0 + 2B(1+\zeta) - 2N_B \ln B$, minimized to give $\hat{B} = N_B$ and 3 equations to solve mutually (and numerically): $\hat{N}_0 = \sum w_i,\;\; 0 = \sum w_i c_i,$ and $\;0 = \sum w_i s_i$, where $w_i = [1 + \hat{q}c_i + \hat{u}s_i + \zeta N_B/\hat{N}_0]^{-1}$

- $\mathrm{MDP}_{99} = \frac{4.29\sqrt{C_S + C_B}}{C_S\sqrt{<\mu_i^2>}}$ for unpolarized data, similar to Elsner+ (2012) result.

Imaging Polarimetry Detector

- Photons ionize atoms in the detector gas
- Direction of the photoelectron is related to the photon's polarization angle
- The electron loses energy through the gas; charge is proportional to E
- Charge drifts to anode





Azimuthal angle

$$\frac{d\sigma}{d\Omega} = f(\zeta) r_0^2 Z^5 \alpha_0^4 \left(\frac{1}{\beta}\right)^{7/2} 4\sqrt{2}\sin^2\theta\cos^2\varphi \text{, where } \beta \equiv \frac{E}{mc^2} = \frac{h\nu}{mc^2}$$

Polar angle

Measuring IXPE Event Tracks

- Current photoelectron track measurement: moments based
- How to do better — full track modeling?

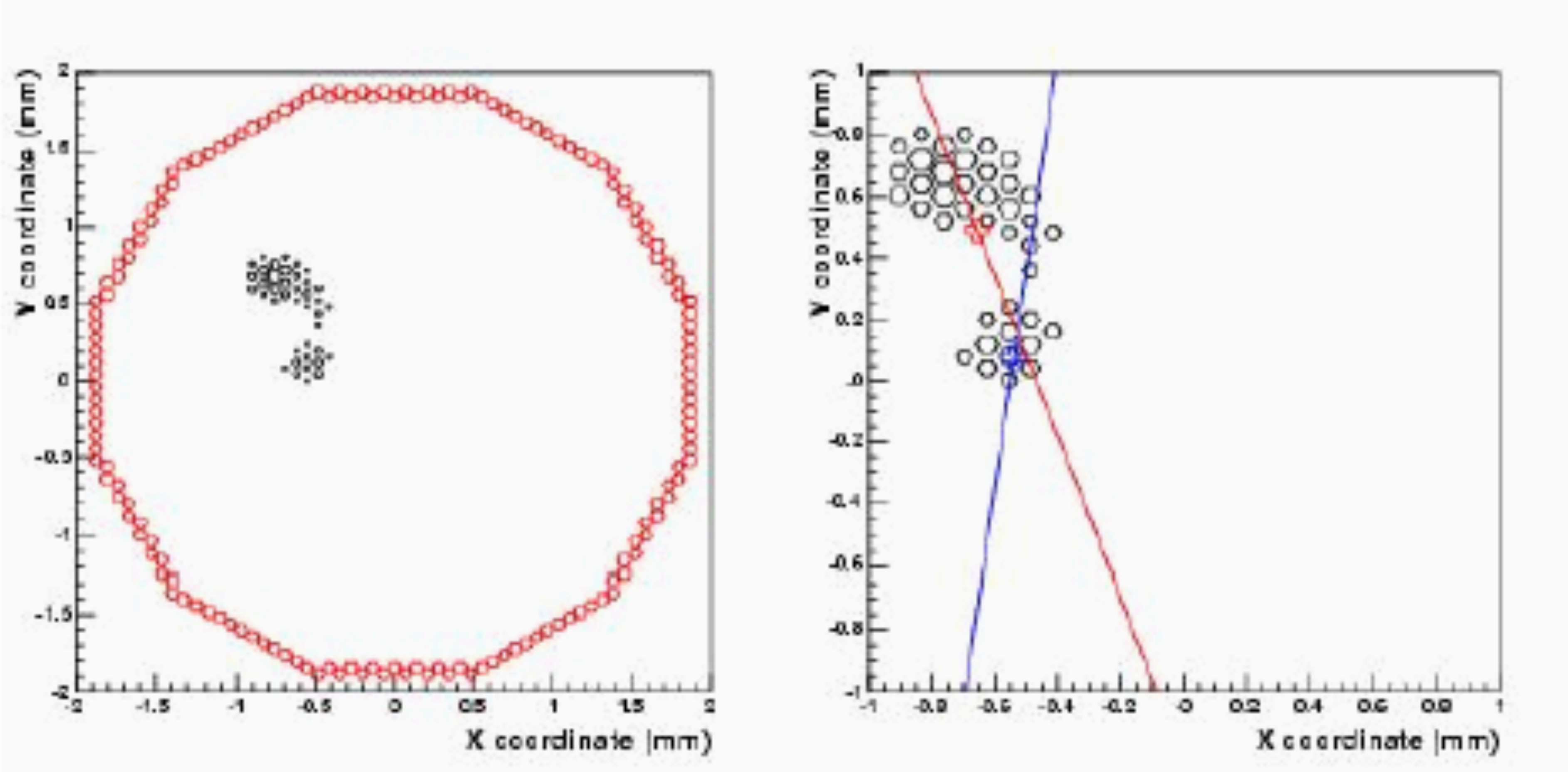


Fig.2. Real track produced in the gas by a 5.9 keV photon. The reconstruction algorithm develops in the following steps: 1) barycenter evaluation of the charge distribution (red cross), 2) reconstruction of the principal axis direction (red line), 3) conversion point evaluation (blue cross), 4) emission direction reconstruction (blue line). The polarization is derived from the photoelectrons angular distribution.

Bellazzini+ '06

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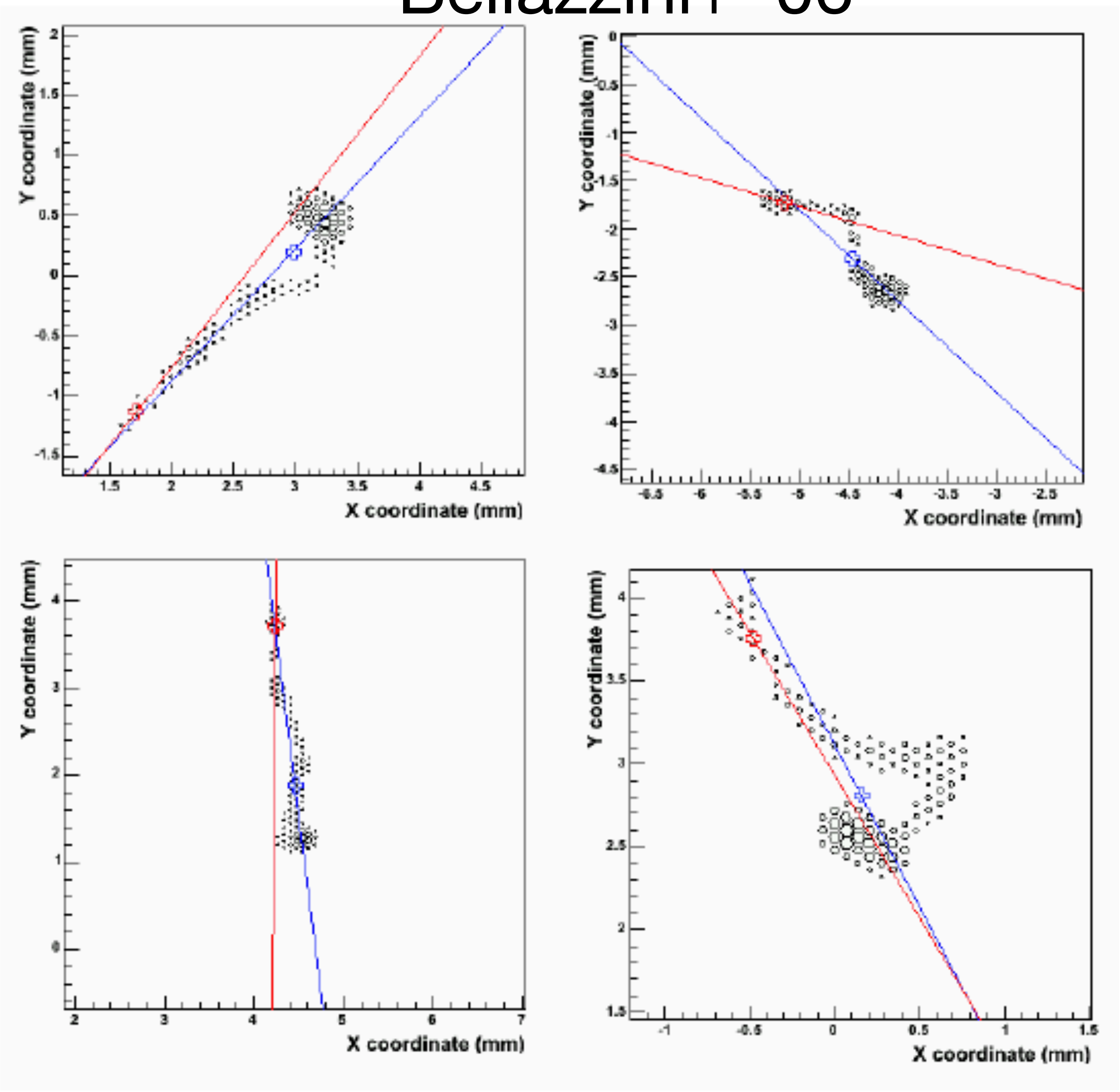
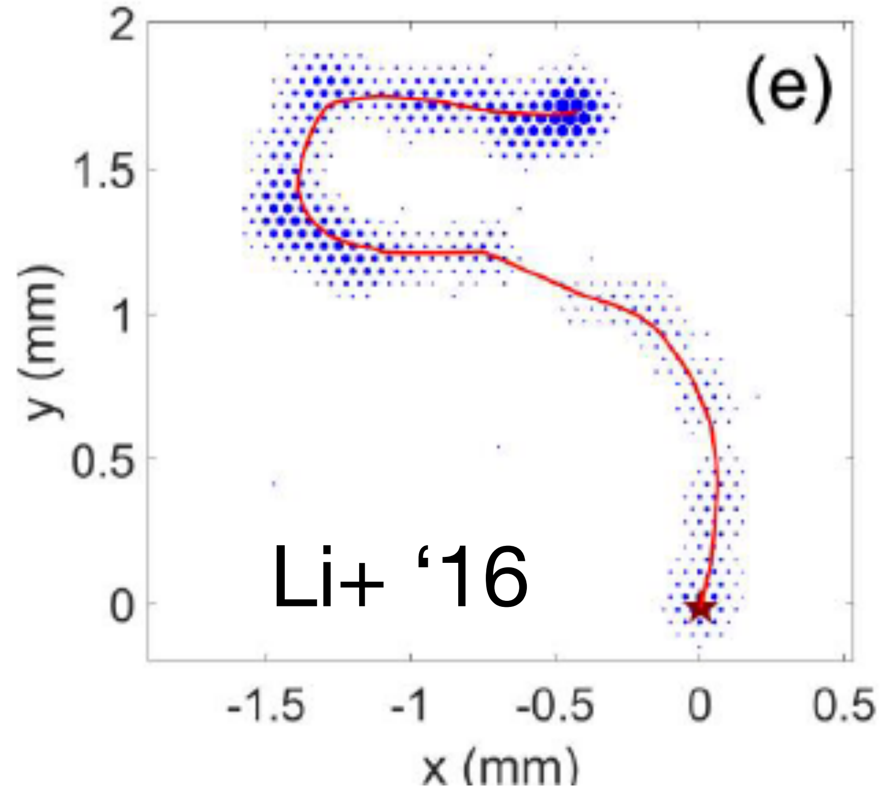
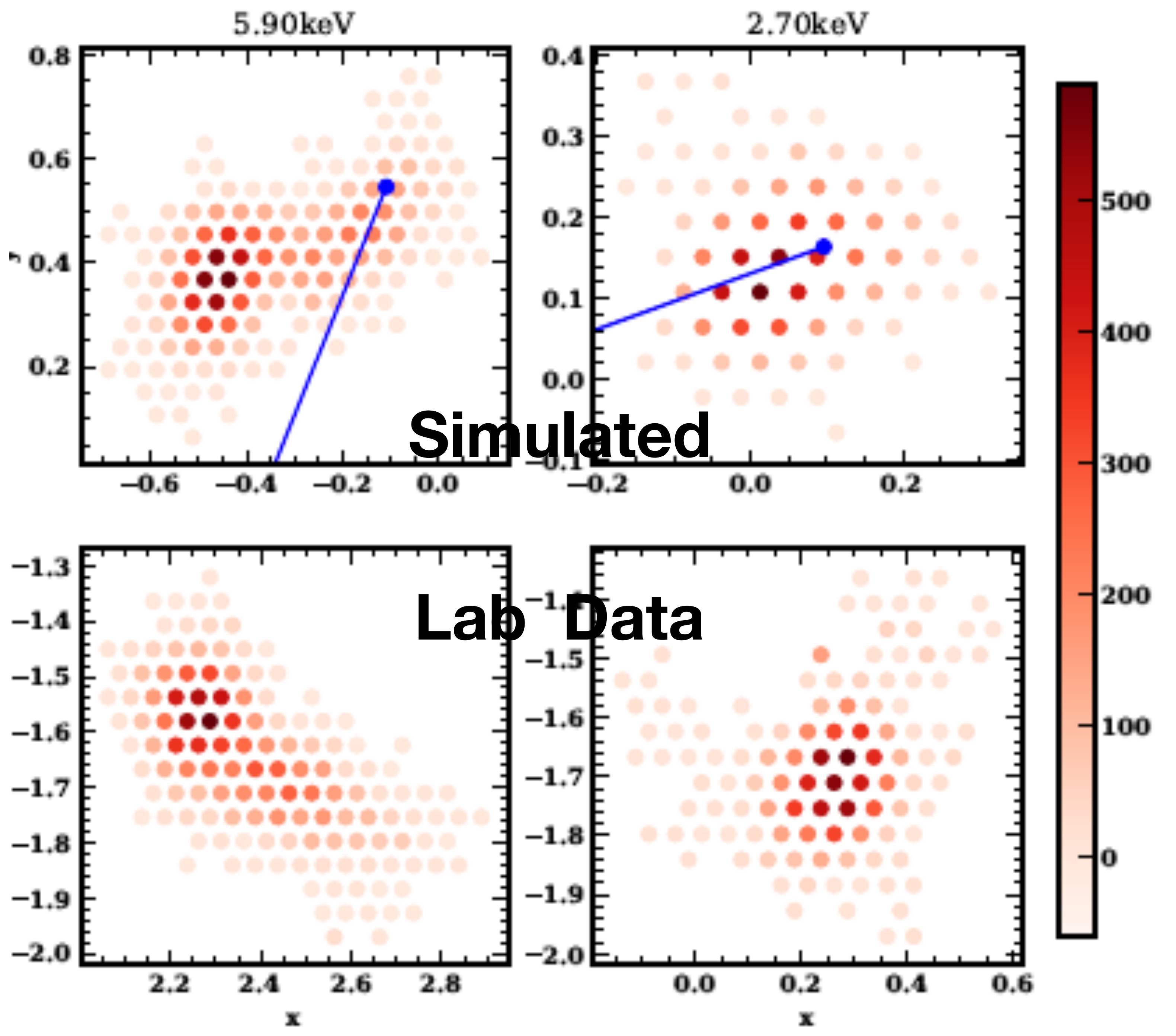
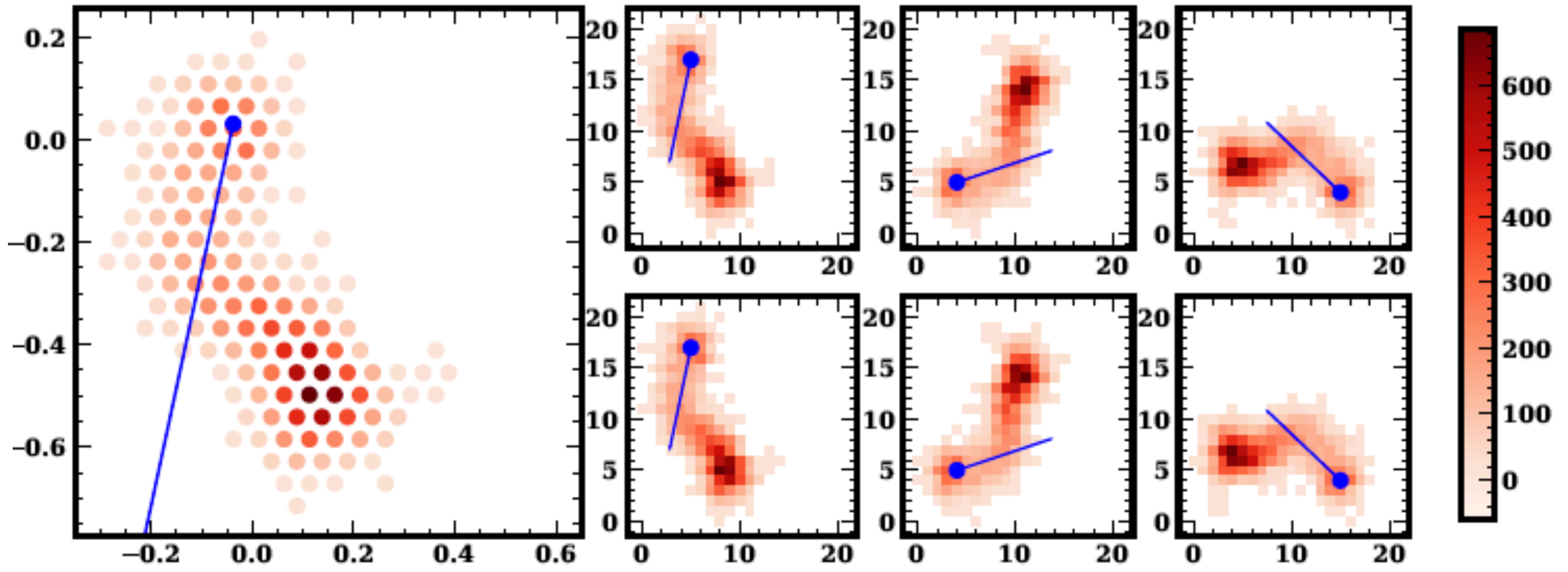




A Neural Network Approach (Peirson+ 2021)

- Convolutional Neural Net (CNN): N events, M networks
- Train to minimize angle errors on simulated data
- <u>Estimate uncertainties in track angles</u>
- Validate on additional simulated data
- Optimize using lab data
- Optimize for best nets using "importance-weighted" likelihood





$$\begin{aligned} \underset{\text{over } \mu,\phi}{\text{minimize}} \quad & -\sum_{j=1}^{M}\sum_{i=1}^{N} \sigma_{ij}^{-\lambda} \log\left(1+\mu\cos\left(2(\theta_{ij}-\phi)\right)\right) \\ \text{subject to} \quad & 0 \le \mu \le 1 \\ & -\pi/2 \le \phi < \pi/2, \end{aligned}$$

Likelihood Analysis Update (Marshall (2021)

- Original: $\lambda(f_E, \Pi, \varphi; E, \psi) = [1 + \Pi\mu_E \cos(2\psi - 2\varphi)]f_E A_E T dE d\psi$
  - gives $\mathrm{MDP}_{99} = 4.29/\sqrt{\sum \mu_E^2 C(E)}$
- Update with:

$$\lambda(f_0, \Pi, \varphi; E, \psi, \sigma) = \int d\psi' \; [1 + \Pi \cos(2\psi' - 2\varphi)]G(\psi; \psi', \sigma) f_E A_E T p(\sigma; E) \quad =$$

$[1 + \Pi e^{-2\sigma^2} \cos(2\psi - 2\varphi)] f_E A_E T \phi(\sigma; E)$ (for Gaussian, also solved for von Mises)

- Transform to σ space: $\tilde{\lambda}(f_0, q, u; \psi, \sigma) = \int \lambda dE = [1 + e^{-2\sigma^2}(q \cos 2\psi + u \sin 2\psi)] f_0 A T \eta(\sigma)$

and $\tilde{S}(q, u) = -2 \sum_i \ln(1 + q e^{-2\sigma_i^2} \cos 2\psi_i + u e^{-2\sigma_i^2} \sin 2\psi_i)$

- Result: $\mathrm{MDP}_{99} = 4.29/\sqrt{\sum e^{-4\sigma^2} C(\sigma)}$, for small Π (or large C[σ])

How do we fit IXPE data?

- Split spectra into I, Q, U
- E is “observed”, E’ is “true” and unknown
- Need RMFs, $R(E', E)$. and ARFs, $\epsilon(E')$
- Assumes mRMF = $\mu(E')R(E', E)$

- Complication: $\mu = f(\alpha)$, α = ellipticity (Di Marco+ 2022)

$$I(E) = \int_{E'} F(E')\epsilon(E')R(E', E)dE' \quad (25)$$

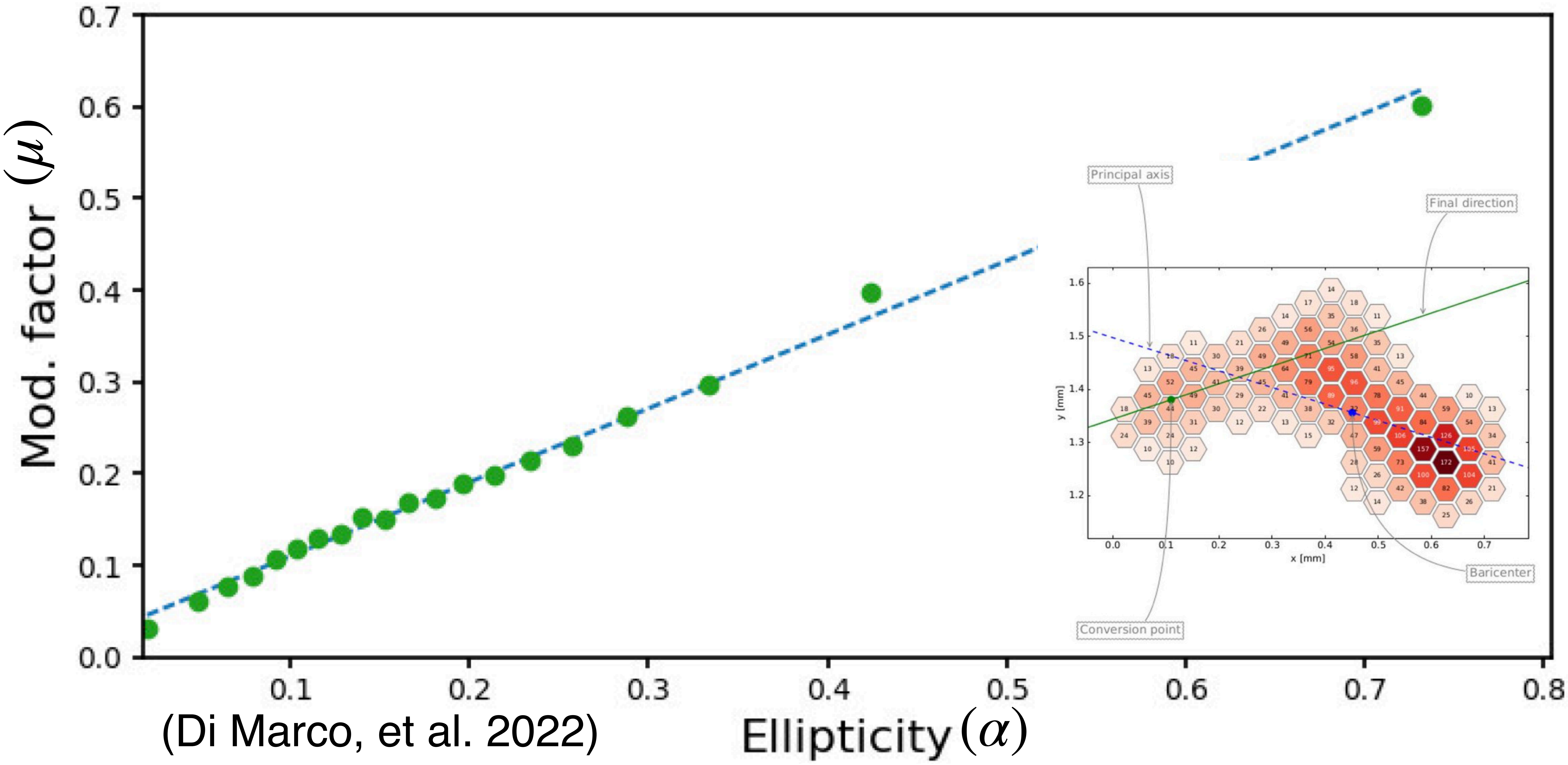
$$U(E) = \int_{E'} W(E')\mu(E')\epsilon(E')R(E', E)dE' \quad (26)$$

$$Q(E) = \int_{E'} Z(E')\mu(E')\epsilon(E')R(E', E)dE' \; . \quad (27)$$

$$O(E, \psi) = I(E) + U(E)\sin(2\psi) + Q(E)\cos(2\psi) \; . \quad (28)$$

Strohmayer (2017)

Inferring the Modulation Factor



(Di Marco, et al. 2022)

IXPE Ellipticity Weighting (Di Marco+ 2022)

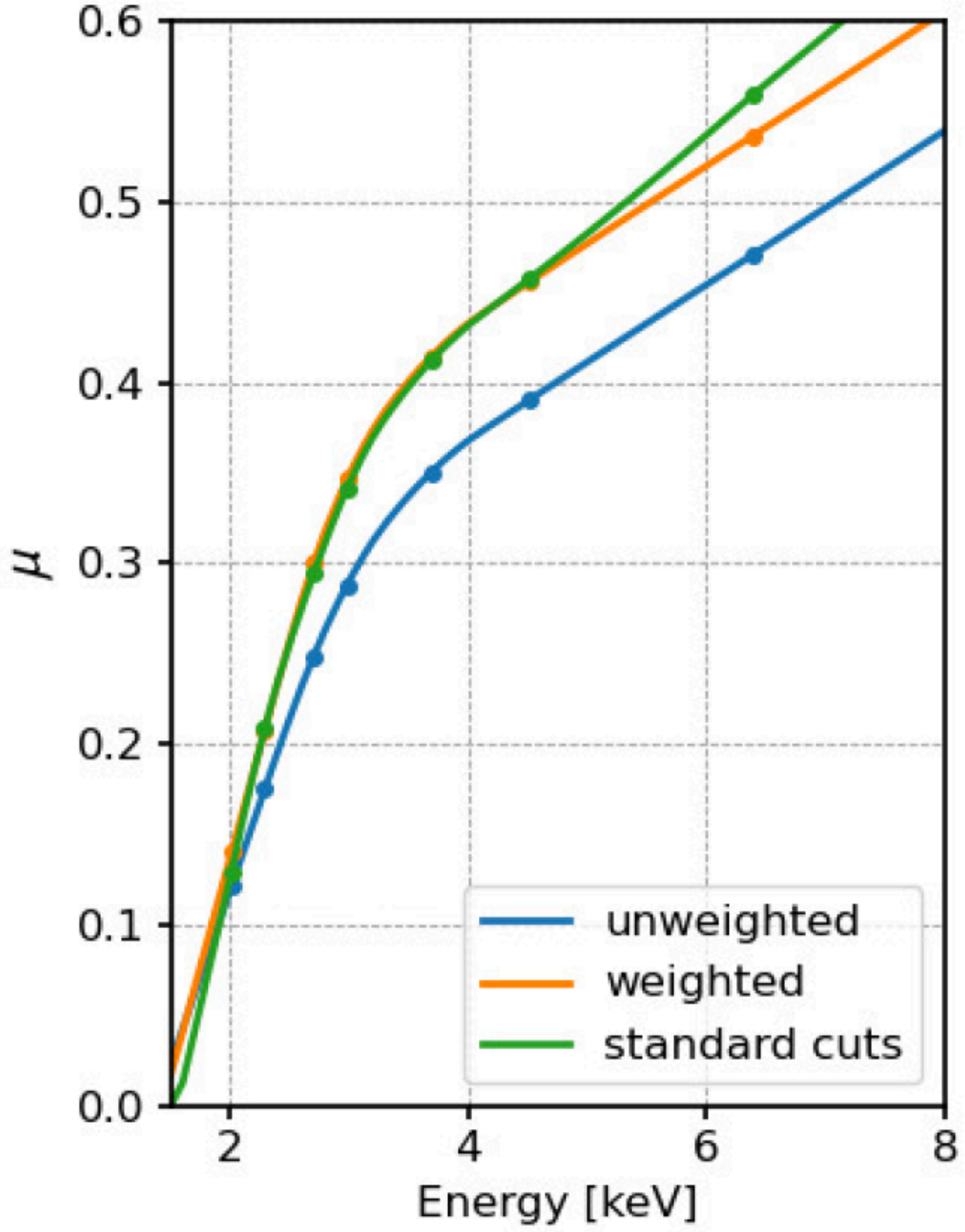
- Compute $w_i = \alpha_i^{0.75}$ for each event
- Estimate I,Q,U:

$$\mathcal{I} = \sum w_i, \quad \mathcal{Q} = 2\sum w_i \cos 2\psi_i, \quad \mathcal{U} = 2\sum w_i \sin 2\psi_i$$

- Compute $\hat{\Pi} = \frac{\sqrt{\mathcal{Q}^2 + \mathcal{U}^2}}{\mu \mathcal{I}}$ with uncertainty

$$\sigma_\Pi^2 \simeq \frac{2\sum w_i^2}{\mu^2 \mathcal{I}^2} = \frac{2}{\mu^2 N_{\rm eff}} \quad (\text{for } \Pi \ll 1)$$

- Develop "weighted" modulation functions
- Weighted MDPs are ~5% better than standard
- Not likelihood based — can do better…



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- Suggestion: mRMF for J (~10) values of α_j
  $\mathscr{M}_j(E', E) = \mu(\alpha_j, E')\epsilon(E')\phi(\alpha_j, E')R(E', E)$

Updating XSPEC analysis

- New model is $Q_j(E,\Theta) = T\int A(E')\mathcal{Q}(E',\Theta)\mathcal{M}_j(E',E)dE', \quad U_j(E,\Theta) = T\int A(E')\mathcal{U}(E',\Theta)\mathcal{M}_j(E',E)dE'$
  - Index j refers to specific values of α_j
  - New detector mRMF is $\mathcal{M}_j(E',E) = \mu(\alpha_j,E')\epsilon(E')\phi(\alpha_j,E')R(E',E)$ New!
  - where $\sum_j \phi(\alpha_j,E') = 1$ and $\sum_j \mu(\alpha_j,E')\phi(\alpha_j,E') = \mu(E')$ (unweighted, uncut)
- Original: $\lambda(f_0,\Pi,\varphi;E,\psi) = [1+\Pi\mu_E\cos(2\psi+2\varphi)]f(E')A(E')TdE'd\psi$
  - gives MDP$_{99}$ $= 4.29/\sqrt{\sum \mu_{E_i}^2 C(E_i)}$
- Now $\lambda(f_0,\Pi,\varphi;E,\alpha_j,\psi) = \int dE'[1+\Pi\mathcal{M}_j(E',E)\cos(2\psi+2\varphi)]f(E')A(E')Td\psi$ and
  $\tilde{S}(q,u) = -2\sum_i \ln[1+q\mu(\alpha_i,E_i)\cos 2\psi_i + u\mu(\alpha_i,E_i)\sin 2\psi_i]$
- and MDP$_{99}$ $= 4.29/\sqrt{\sum_j\sum_i \mu_j(E_i)^2 C(E_i,\alpha_j)}$

Azimuthally Nonuniform Response

- Define $w(\psi)$ as the relative exposure to angle ψ, s.t. $\int w(\psi)d\psi = 1$
  - If uniform (like IXPE), $w(\psi) = 1/(2\pi)$
  - For a Bragg reflector, where $w(\psi) = \delta(\psi - \psi_0)$ or $w(\psi) = \frac{1}{n_B}\sum_{i=1}^{n_B}\delta(\psi - \psi_i)$ for n_B reflectors
- Count density: $\lambda(\psi) = [1 + \mu_E(q\cos 2\psi + u\sin 2\psi)]n_E A_E T w(\psi)d\psi$
- Likelihood: $S(f_0, q, u) = -2N\ln f_0 - 2\sum_i \ln(1 + q\mu_i\alpha_i + u\mu_i\beta_i) + 2Kf_0 + 2K_\mu f_0 Aq + 2K_\mu f_0 Bu$, where

  $\alpha(\psi) \equiv w(\psi)\cos 2\psi$, $\beta(\psi) \equiv w(\psi)\sin 2\psi$, $A \equiv \int \alpha(\psi)d\psi$, and $B \equiv \int \beta(\psi)d\psi$, and

  $K = \frac{T}{f_0}\int f(E;\xi)A_E dE$ and $K_\mu = \frac{T}{f_0}\int f(E;\xi)A_E\mu_E dE$ are treated as uninteresting constants
- If $w(\psi) \neq f(\psi)$ or $\psi_i = \psi_0 + \frac{\pi i}{n_B}$ (so $A = B = 0$) or if $q = u = 0$, then $n_0 = N$
- All parameters are covariant otherwise

Azimuthally Nonuniform Response II

- Minimizing the log likelihood gives $\hat{f}_0 = \dfrac{N}{K + K_\mu(A\hat{q} + B\hat{u})}$,

  $AK_\mu \hat{f}_0 = \sum_i W_i \mu_i \alpha_i$ and $BK_\mu \hat{f}_0 = \sum_i W_i \mu_i \beta_i$, where $W_i \equiv (1 + \hat{q}\mu_i\alpha_i + \hat{u}\mu_i\beta_i)^{-1}$

- Transform to (Π, φ) space:

  $S(\hat{f}_0, \Pi, \phi) = 2N \ln[K + K_\mu \Pi(A\cos 2\phi + B \sin 2\phi)] - 2\sum_i \ln[1 + \Pi w_i \mu_i(\cos 2\psi_i \cos 2\phi + \sin 2\psi_i \sin 2\phi)]$

- As $\Pi \to 0, \dfrac{1}{\sigma_\Pi^2} \approx \sum_i w_i^2 \mu_i^2 \cos^2(2\psi_i - 2\varphi) - NK_\mu^2(A\cos 2\varphi + B\sin 2\varphi)^2/K^2$

  "usual" **"watch out!"**

- Thus, want A=B=0 as much as possible!

Summary

- Polarimetry adds complexity to spectral fits
- xspec: updated for simple case
- Real IXPE data require mRMFs in E and ellipticity α
- Accounting for background is a simple addition
- Important to design instruments with uniform $w(\psi)$