Double Lomb-Scargle Periodogram: Super Resolution

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Javier Pascual Granado, Tony Readhead, Francisco De Geronimo

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Practical Time Series Data Analysis:

Lessons from Time-Domain Astronomy

SHAMELESS COMMERCE DIVISION:

CAMBRIDGE OBSERVING HANDBOOKS

FOR RESEARCH ASTRONOMERS (TO APPEAR)

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To every child who wondered
what lies beyond the blue sky
and then grew up finding out.

What's New in the Handbook?

- Bayesian Block Histograms Allowing Empty Bins
- Autoregressive Moving Average
  - The ARMA Convolution Group
  - The Arrow of Time
- "Lomb-Scargle Fourier Transform"
- Double Periodograms
- Omnigrams

The Hunt for Periodicity: Rotation … Oscillation … Revolution … Waves

The PERIODOGRAM (Schuster, 1898) — power spectrum estimate to "find hidden periodocities"

Problems:

Statistical: Noise does not get smaller as N increases
MANY confusing peaks in the periodogram (~N/6?)
Colored noise background thwarts significance analysis
Spectral leakage: corrupts the periodogram
Uneven sampling …

Subjective: Trials factors
Exploratory vs. confirmatory settings
Hypothesis framing
Cherry picking
Systematics
Black box analysis

$$CC = \sum w_n cos^2(\omega t_n)$$
$$SS = \sum w_n sin^2(\omega t_n)$$
$$XC = \sum w_n x_n cos(\omega t_n)$$
$$XS = \sum w_n x_n sin(\omega t_n)$$
$$CS = \sum w_n cos(\omega t_n)\ sin(\omega t_n)$$

$$y_n = a\ cos(\omega t_n) + b\ sin(\omega_1 t_n)$$

$$\log L = -\frac{1}{2} \sum_{n=1}^{N} (\frac{x_n - y_n}{\sigma_n})^2\ ,$$

$$R = a^2\ CC + b^2\ SS + 2(ab\ CS - a\ XC - b\ XS)$$

Least Squares ≈ Maximum Likelihood

The Lomb Trick:
Trade CS term for phase term

$$CC = \sum w_n cos^2(\omega t_n - \theta)$$
$$SS = \sum w_n sin^2(\omega t_n - \theta)$$
$$XC = \sum w_n x_n cos(\omega t_n - \theta)$$
$$XS = \sum w_n x_n sin(\omega t_n - \theta)$$
$$CS = \sum w_n cos(\omega t_n - \theta) sin(\omega t_n - \theta) = 0$$

```
P = XC^2 / CC + XS^2 / SS

a = ( SS*XC - CS*XS ) / (CC*SS - CS^2)
b = ( CC*XS - CS*XC ) / (CC*SS - CS^2)
```

Schuster-like Maximum Likelihood Periodogram

$$P_{\text{Schuster}}(\omega) = |\sum x_n e^{-i\omega t_n}\ |^2$$
$$= (\sum x_n cos\ \omega t_n)^2 + (\sum x_n sin\ \omega t_n)^2$$

```
P(Lomb-Scargle) = XC^2 / CC + XS^2 / SS

a = XC/CC
b = XS/SS
```

The Lomb-Scargle Periodogram

* Any time series data — measurements, events (photons), etc.
* Any sampling in time — random, systematic, gaps, jitter …
* Easy application of statistical weights

* Window function
  - Different from sinc**2 if sampling not even
  - Does not correct for the sampling; deconvolve the window in post
  - Application of (multi-)tapers tames the spectral leakage

* Derivable from a Fourier transform for arbitrarily sampled data
  - LSP = | FT | **2
  - Complex FT enables cross-correlation functions, cross-spectra, co-spectra
  - Invertible, with high frequencies (large effective Nyquist frequency)

Fourier Transform of Unevenly Spaced Time Series

LSP can be derived from this Fourier Transform = | FT | ^{2}

Weights

Measured
amplitudes

Frequencies
(arbitrary)

time tags

$$FT_X(\omega) = F_0 \left\{ \frac{\sum_n w_n x_n \cos \omega(\ t_n - \tau(\omega)\ )}{[\ \sum_m w_m \cos^2 \omega(\ t_m - \tau(\omega)\ )\ ]^{1/2}} + i\, \frac{\sum_n w_n x_n \sin \omega(\ t_n - \tau(\omega)\ )}{[\ \sum_m w_m \sin^2 \omega(\ t_m - \tau(\omega)\ )]^{1/2}} \right\} \tag{1}$$

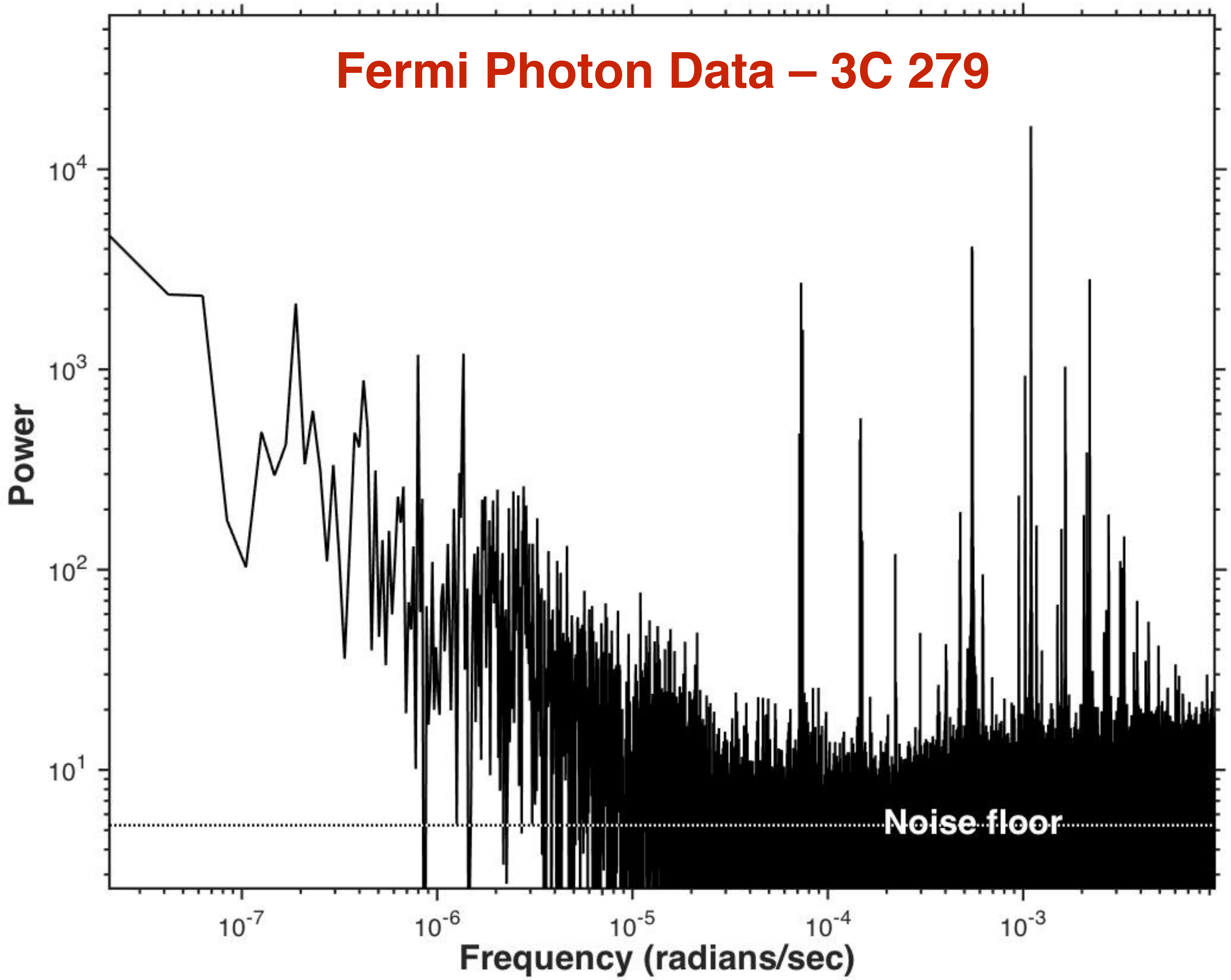
Lomb shift

Weights

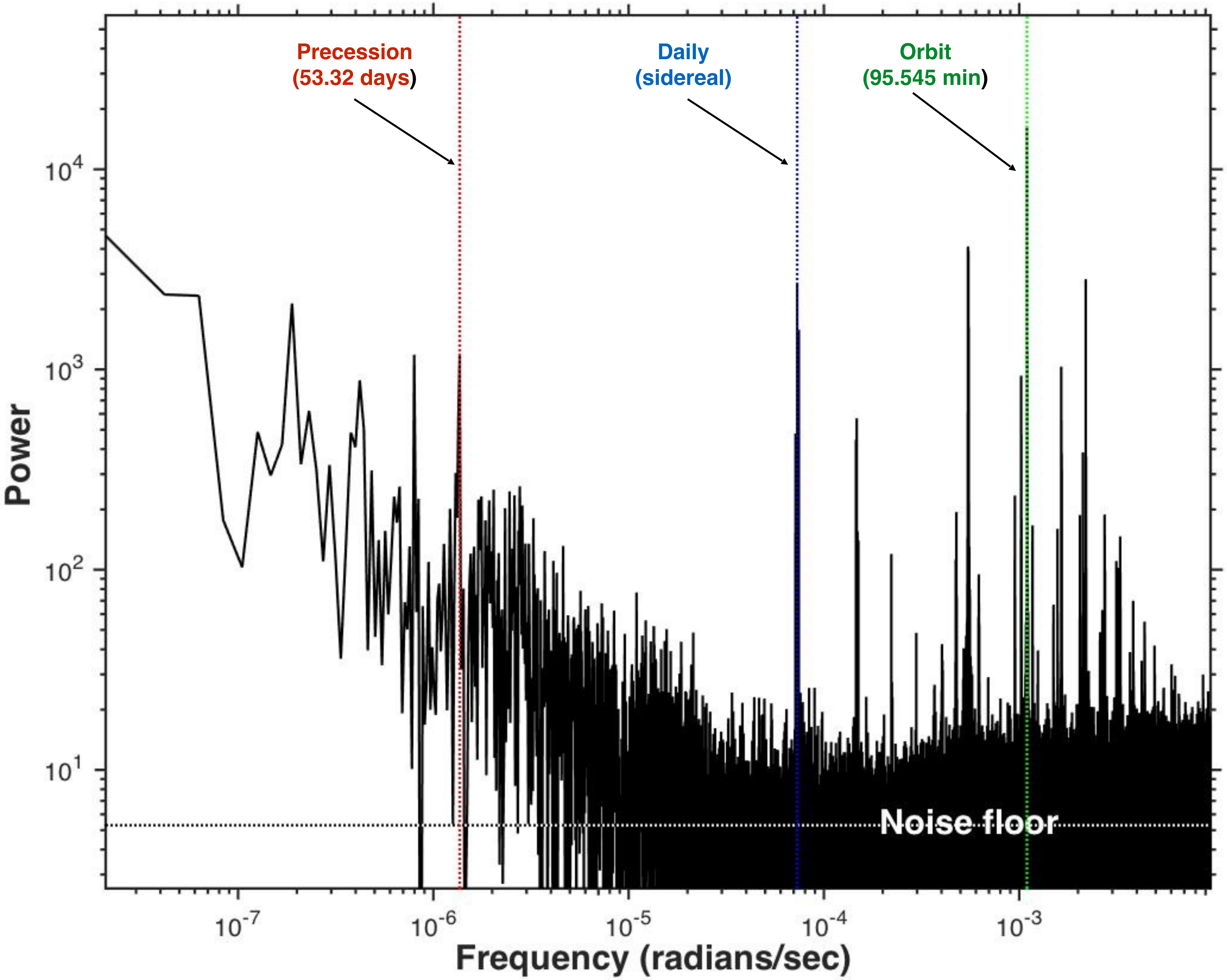
where

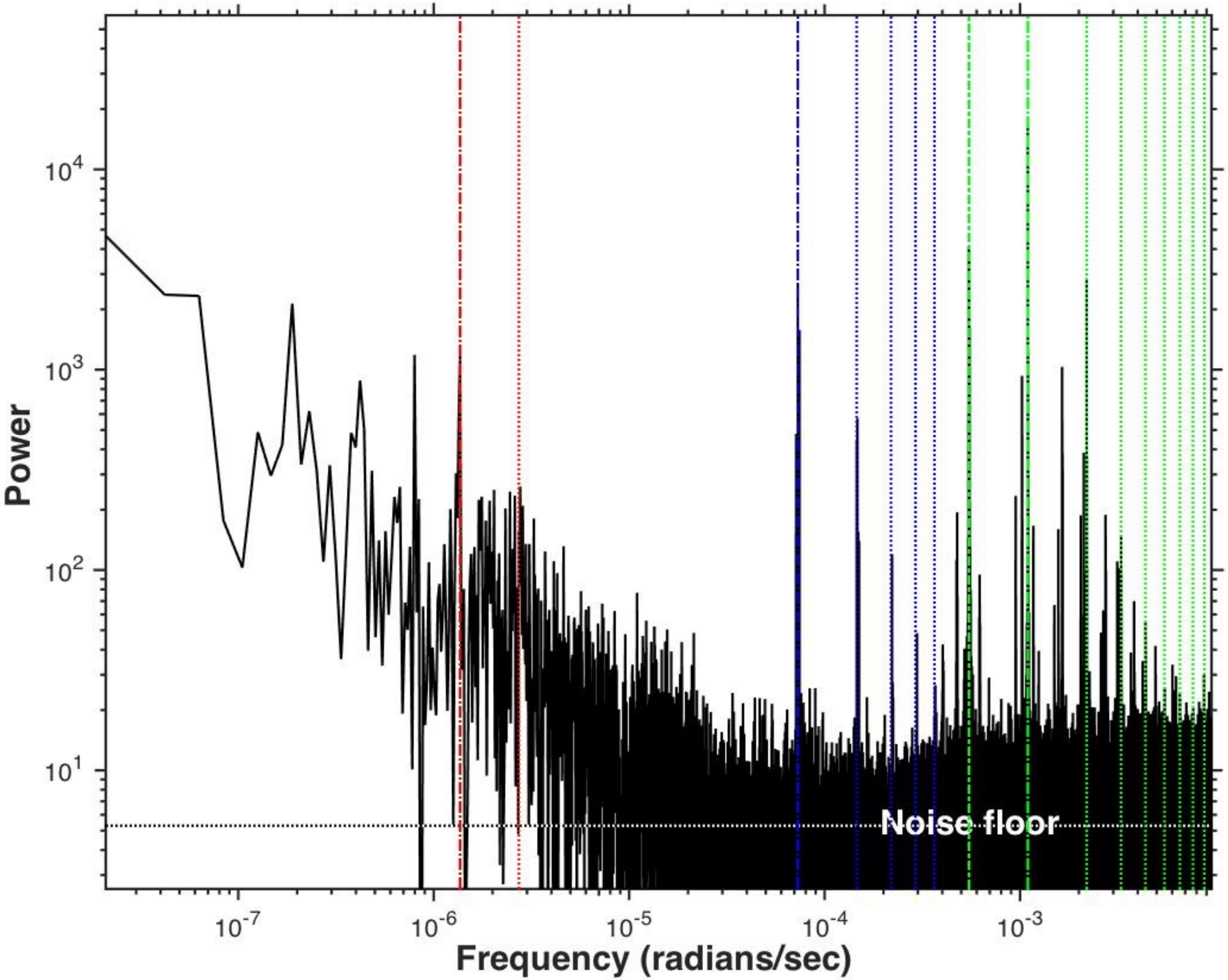
$$\tau(\omega) = \frac{1}{2\omega} \arctan[\ (\sum w_n \sin 2\omega t_n)\ /\ (\sum w_m \cos 2\omega t_m)\ ] \tag{2}$$

where all sums are over $n = 1, 2, ..., N$.

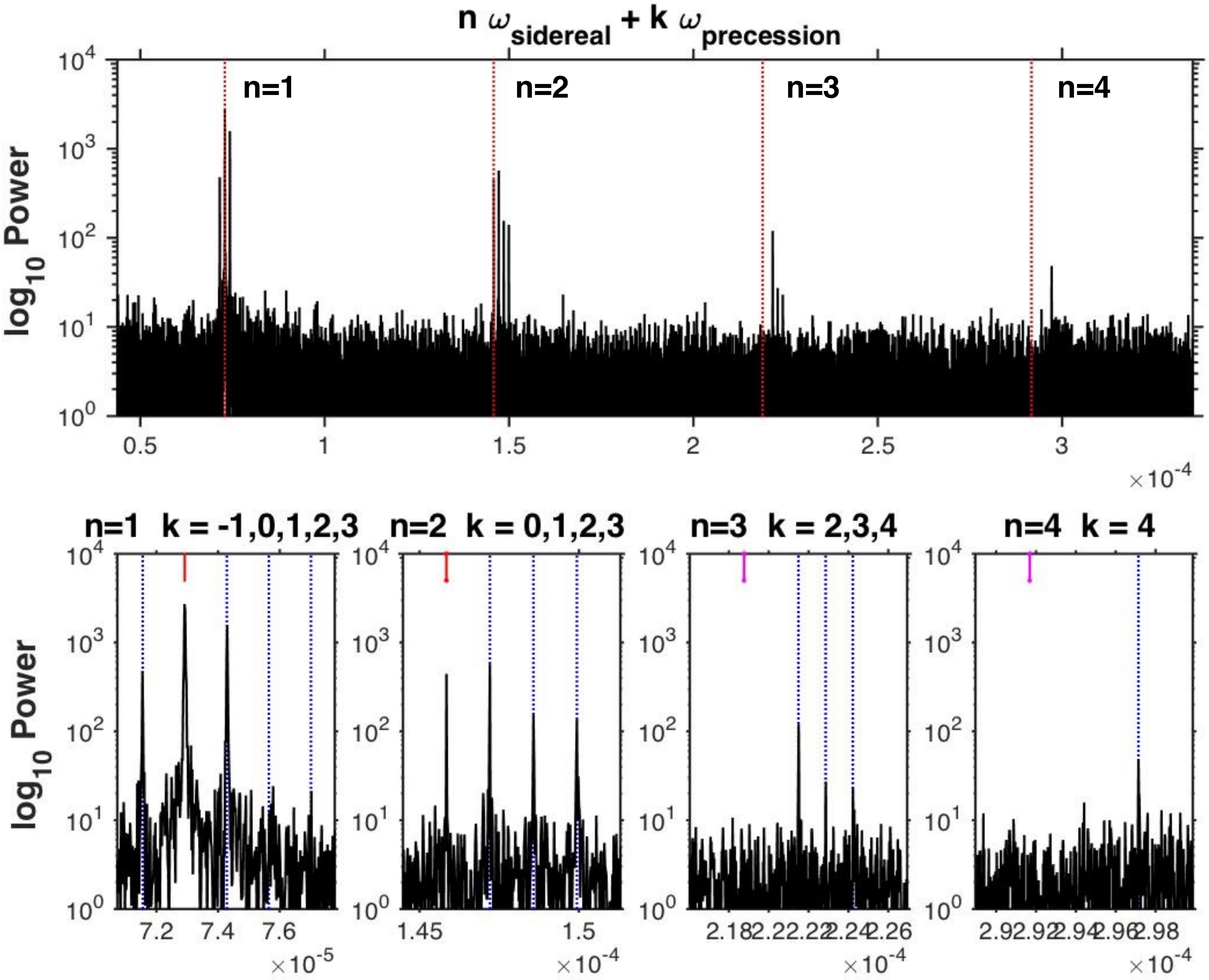


Fermi Photon Data – 3C 279
Power
10^4
10^3
10^2
10^1
Noise floor
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
Frequency (radians/sec)

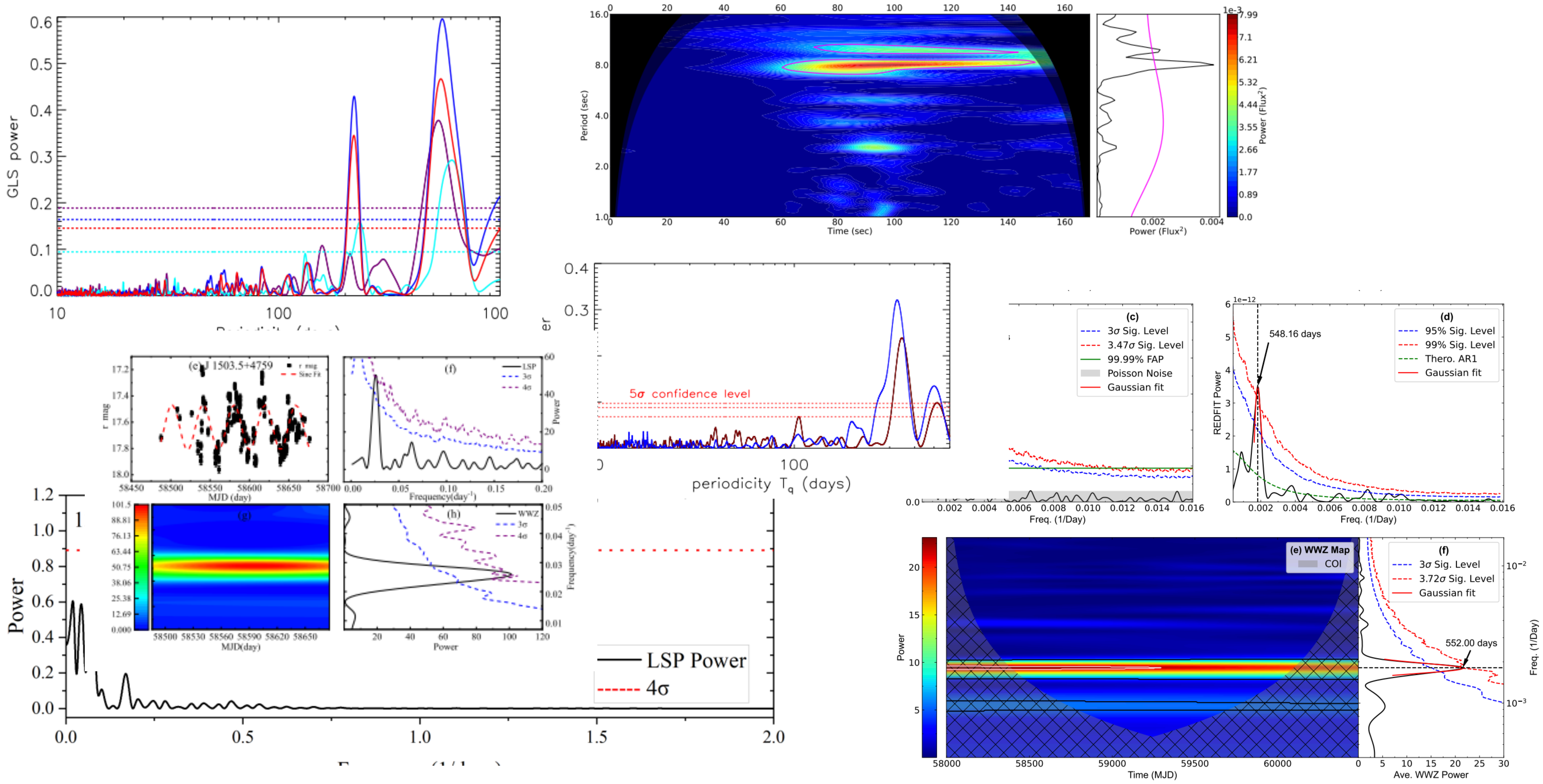

Precession
(53.32 days)
Daily
(sidereal)
Orbit
(95.545 min)
Power
10^4
10^3
10^2
10^1
Noise floor
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
Frequency (radians/sec)



Power
Frequency (radians/sec)
Noise floor
10^4
10^3
10^2
10^1
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}

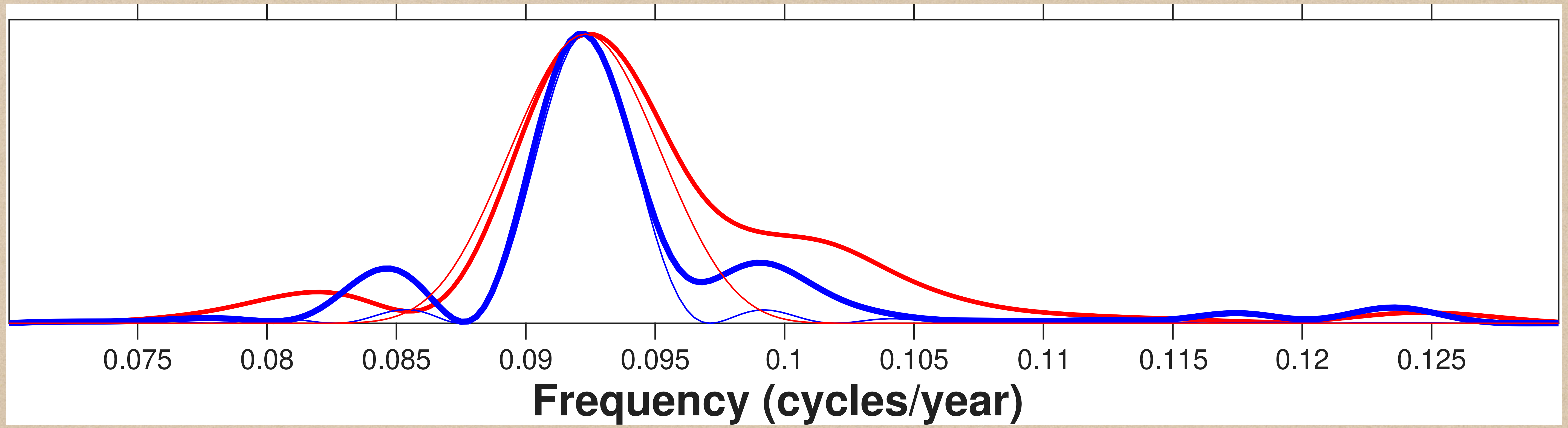


n ω sidereal + k ω precession
n=1
n=2
n=3
n=4
log10 Power
×10^-4
n=1 k = -1,0,1,2,3
n=2 k = 0,1,2,3
n=3 k = 2,3,4
n=4 k = 4
×10^-5



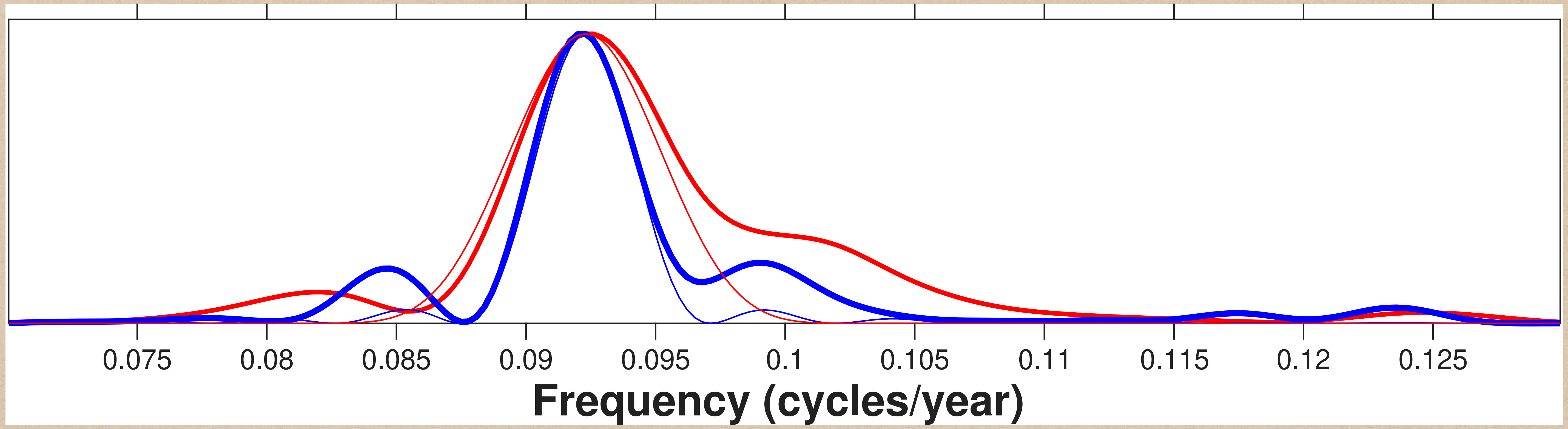
GLS power
Periodicity (days)
Period (sec)
Time (sec)
Power (Flux^2)
5σ confidence level
periodicity T_q (days)
(c)
3σ Sig. Level
3.47σ Sig. Level
99.99% FAP
Poisson Noise
Gaussian fit
Freq. (1/Day)
(d)
95% Sig. Level
99% Sig. Level
Thero. AR1
Gaussian fit
548.16 days
REDFIT Power
(e) J 1503.5+4759
r mag
Sine Fit
MJD (day)
(f)
LSP
3σ
4σ
Frequency(day^{-1})
(g)
MJD(day)
(h)
WWZ
Power
LSP Power
4σ
Frequency (1/day)
(e) WWZ Map
COI
(f)
3σ Sig. Level
3.72σ Sig. Level
Gaussian fit
552.00 days
Time (MJD)
Ave. WWZ Power
Freq. (1/Day)

LSP of Sunspot Number Time Series



- Frequency Uncertainty ≈ width of peak
- Best Resolution ≈ width of peak

LSP of Sunspot Number Time Series



- Frequency (U[illegible]y ≈ width of peak
- Best Res[illegible] width of peak

Wrong!

Generalizations of The Lomb-Scargle Periodogram

* Non-sinusoidal periodic signals
* Damped sinusoids (amplitude modulation)
* Additive Constant (“GLSP”)
* Multivariate
* Tapered, Stacked, etc.
* Basis functions other than sinusoids
* Multi-frequency (Barning 1963 + more …)

Criticisms of The Lomb-Scargle Periodogram

* Does not include weights
* Window function complicated
* Spectral Leakage
* Nyquist Frequency not Defined
* Restricted model: not good for non-white noise
* Restricted model: single-frequency sinusoid

Criticisms of The Lomb-Scargle Periodogram

* Does not include weights. Yes it does! (Gilliland and Baliunas, Scargle)
* Window function complicated This was not promised; deconvolve it yourself.
* Spectral Leakage LSP can reduce aliasing and other leakage.
* Nyquist Frequency not Defined. Define your own! (vanderPlas)
* Restricted model: not good for non-white noise. Try it, you'll like it!
* Restricted model: single-frequency sinusoid. The rest of this talk …

$$\log L = -\frac{1}{2}\sum_{n=1}^{N}\left(\frac{x_n - y_n}{\sigma_n}\right)^2$$

$$\begin{aligned} y_n =& a_1\ cos(\omega_1 t_n - \theta) + b_1\ sin(\omega_1 t_n - \theta) \\ &+ a_2\ cos(\omega_2 t_n - \theta) + b_2\ sin(\omega_2 t_n - \theta)\ , \end{aligned}$$

Least-Squares

$$\begin{aligned} R = & a_1^2\ CC11 + b_1^2\ SS11 + a_2^2\ CC22 + b_2^2\ SS22 \\ & - 2(a_1 XC1 + b_1 XS1 + a_2 XC2 + b_2 XS2) \\ & + 2(a_1 b_1 CS11 + a_1 a_2 CC12 + a_1 b_2 CS12 \\ & + b_1 a_2 SC12 + b_1 b_2 SS12 + a_2 b_2 CS22\ ) \end{aligned}$$

$$\begin{aligned} CC11 &= \sum w_n cos^2(\Omega_1) \\ SS11 &= \sum w_n sin^2(\Omega_1) \\ CC22 &= \sum w_n cos^2(\Omega_2) \\ SS22 &= \sum w_n sin^2(\Omega_2) \\ XC1 &= \sum w_n x_n cos(\Omega_1) \\ XS1 &= \sum w_n x_n sin(\Omega_1) \\ XC2 &= \sum w_n x_n cos(\Omega_2) \\ XS2 &= \sum w_n x_n sin(\Omega_2) \\ CC12 &= \sum w_n cos(\Omega_1) cos(\Omega_2) \\ CS12 &= \sum w_n cos(\Omega_1) sin(\Omega_2) \\ SC12 &= \sum w_n sin(\Omega_1) cos(\Omega_2) \\ SS12 &= \sum w_n sin(\Omega_1) sin(\Omega_2) \\ CS11 &= \sum w_n cos(\Omega_1) sin(\Omega_1) \\ CS22 &= \sum w_n cos(\Omega_2) sin(\Omega_2) \end{aligned}$$

Bayesian Marginalized Posterior

$$P(\omega_1, \omega_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(a_1, b_1, a_2, b_2)\, da_1 db_1 da_2 db_2\ .$$

$$\int_{-\infty}^{\infty} e^{-(Ax^2+Bx+C)}\ dx = \sqrt{\frac{\pi}{A}} e^{(B^2/4A)-C}\ .$$

$$\begin{aligned} & RC = \text{coeffs}(R, a1) \\ & A = RC(3) \\ & B = RC(2) \\ & C = RC(1) \\ & R = -(B^2/(4A)) + C \end{aligned}$$

Other basis functions: just plug in!

$$\begin{aligned} y_n = & a_1\ cos((\omega_1+\omega_2)t_n)\ cos((\omega_1-\omega_2)t_n) \\ & + b_1\ sin((\omega_1+\omega_2)t_n)\ cos((\omega_1-\omega_2)t_n) \\ & + a_2\ cos((\omega_1+\omega_2)t_n)\ sin((\omega_1-\omega_2)t_n)/(\omega_2-\omega_1) \\ & + b_2\ sin((\omega_1+\omega_2)t_n)\ sin((\omega_1-\omega_2)t_n)/(\omega_2-\omega_1) \end{aligned}$$

Baluev

$$y_n = \sum_{j=1}^{m} \mathbf{B_j G_j(t_n, \hat{\omega}_j)} + \epsilon_n$$

$$-\log L = \frac{1}{2}\sum_{n=1}^{N}\left(\frac{\sum_{j=1}^{m}(x_n - B_j G_j(t_n,\hat{\omega}_j))}{\sigma_n}\right)^2 \quad ,$$

Absorb errors into definitions of x and G:

$$\sum_{n=1}^{N} x_n^2 - 2\sum_{j=1}^{m} B_j \sum_{n=1}^{N} x_n G_j(t_n,\hat{\omega}_j) \ + \sum_{n=1}^{N}\left(\sum_{j=1}^{m} B_j G_j(t_n,\hat{\omega}_j) \ \right)^2$$

Find the B's given arbitrary G's
Then plug into the Residual Formula to get the Omnigram

```
cos_11_vec = cos( ww_11 * tt_vec )
 sin_11_vec = sin( ww_11 * tt_vec )

gg11 = sum( wt_vec * abs( cos_11_vec ).^ 2 )
gg22 = sum( wt_vec * abs( sin_11_vec ).^ 2 )
gg12 = sum( wt_vec * cos_11_vec .* sin_11_vec )
dG1  = sum( wt_vec * xx_vec .* cos_11_vec )
dG2  = sum( wt_vec * xx_vec .* sin_11_vec )

cos_22_vec = cos( ww_22 * tt_vec )
 sin_22_vec = sin( ww_22 * tt_vec )

 gg33 = sum( wt_vec * abs( cos_22_vec ).^ 2)
 gg44 = sum( wt_vec * abs( sin_22_vec ).^ 2)
 gg34 = sum( wt_vec * cos_22_vec .* sin_22_vec )
 dG3  = sum( wt_vec * xx_vec .* cos_22_vec )
 dG4  = sum( wt_vec * xx_vec .* sin_22_vec )

gg13 =  sum( wt_vec * cos_11_vec .*cos_22_vec )
gg24 =  sum( wt_vec * sin_11_vec .* sin_22_vec )
gg14 =  sum( wt_vec * cos_11_vec .* sin_22_vec )
gg23 =  sum( wt_vec * sin_11_vec .* cos_22_vec )
```

```
Q_dG1 =  gg22*gg34^2 +gg24^2*gg33 +gg23^2*gg44 -2*gg23*gg24*gg34 -gg22*gg33*gg44
Q_dG2 =  gg13*gg24*gg34 -gg12*gg34^2 +gg14*gg23*gg34 -gg14*gg24*gg33 -gg13*gg23*gg44 +gg12*gg33*gg44
Q_dG3 =-(gg13*gg24^2 -gg14*gg23*gg24 -gg12*gg24*gg34 +g14*gg22*gg34 +gg12*gg23*gg44 -gg13*gg22*gg44 )
Q_dG4 =  gg13*gg23*gg24 -gg14*gg23^2 +gg12*gg23*gg34 -gg12*gg24*gg33 -gg13*gg22*gg34 +gg14*gg22*gg33

        B1 = (Q_dG1*dG1 + Q_dG2*dG2 + Q_dG3*dG3 + Q_dG4*dG4) / Denominator
…
                Denominator = ...
                - gg12^2*gg34^2 + gg33*gg44*gg12^2 - 2*gg44*gg12*gg13*gg23
                + 2*gg12*gg13*gg24*gg34 + 2*gg12*gg14*gg23*gg34
                - 2*gg33*gg12*gg14*gg24 - gg13^2*gg24^2 + gg22*gg44*gg13^2
                + 2*gg13*gg14*gg23*gg24 - 2*gg22*gg13*gg14*gg34
                - gg14^2*gg23^2 + gg22*gg33*gg14^2 + gg11*gg44*gg23^2
                - 2*gg11*gg23*gg24*gg34 + gg11*gg33*gg24^2
                  + gg11*gg22*gg34^2 - gg11*gg22*gg33*gg44


             DP = gg11*B1^2 + 2*gg12*B1*B2 + 2*gg13*B1*B3 + 2*gg14*B1*B4
               - 2*dG1*B1 + gg22*B2^2 + 2*gg23*B2*B3 + 2*gg24*B2*B4 - 2*dG2*B2
                + gg33*B3^2 + 2*gg34*B3*B4 - 2*dG3*B3 + gg44*B4^2 - 2*dG4*B4
```

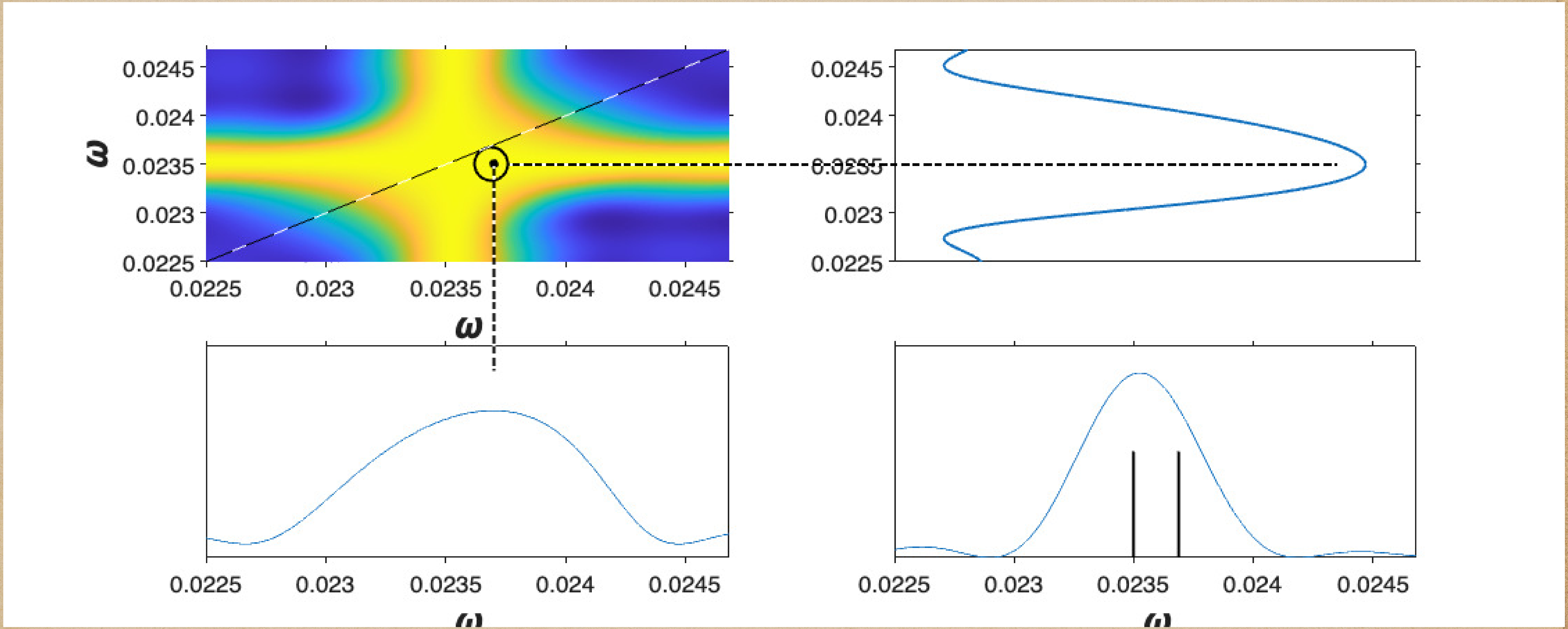
```
Bayes Numerator = ...
            - 2*CC12^2*SS12*XXS1*XXS2 + SS22*CC12^2*XXS1^2
            + SS11*CC12^2*XXS2^2 + 2*CC12*CS11*CS22*XXS1*XXS2
            - 2*CC12*CS11*SC12*XXS2^2 + 2*CC12*CS11*SS12*XXC2*XXS2
            - 2*SS22*CC12*CS11*XXC2*XXS1 - 2*CC12*CS12*CS22*XXS1^2
            + 2*CC12*CS12*SC12*XXS1*XXS2 + 2*CC12*CS12*SS12*XXC2*XXS1
            - 2*SS11*CC12*CS12*XXC2*XXS2 + 2*CC12*CS22*SS12*XXC1*XXS1
            - 2*SS11*CC12*CS22*XXC1*XXS2 + 2*CC12*SC12*SS12*XXC1*XXS2
            - 2*SS22*CC12*SC12*XXC1*XXS1 - 2*CC12*SS12^2*XXC1*XXC2
            + 2*SS11*SS22*CC12*XXC1*XXC2 - 2*CS11^2*CS22*XXC2*XXS2
            + SS22*CS11^2*XXC2^2 + CC22*CS11^2*XXS2^2
            + 2*CS11*CS12*CS22*XXC2*XXS1 + 2*CS11*CS12*SC12*XXC2*XXS2
            - 2*CS11*CS12*SS12*XXC2^2 - 2*CC22*CS11*CS12*XXS1*XXS2
            - 2*CS11*CS22^2*XXC1*XXS1 + 2*CS11*CS22*SC12*XXC1*XXS2
            + 2*CS11*CS22*SS12*XXC1*XXC2 - 2*SS22*CS11*SC12*XXC1*XXC2
            - 2*CC22*CS11*SS12*XXC1*XXS2 + 2*CC22*SS22*CS11*XXC1*XXS1
            - 2*CS12^2*SC12*XXC2*XXS1 + SS11*CS12^2*XXC2^2
            + CC22*CS12^2*XXS1^2 + 2*CS12*CS22*SC12*XXC1*XXS1
            - 2*SS11*CS12*CS22*XXC1*XXC2 - 2*CS12*SC12^2*XXC1*XXS2
            + 2*CS12*SC12*SS12*XXC1*XXC2 - 2*CC22*CS12*SS12*XXC1*XXS1
            + 2*CC22*SS11*CS12*XXC1*XXS2 + SS11*CS22^2*XXC1^2
            + CC11*CS22^2*XXS1^2 - 2*CS22*SC12*SS12*XXC1^2
            - 2*CC11*CS22*SC12*XXS1*XXS2 - 2*CC11*CS22*SS12*XXC2*XXS1
            + 2*CC11*SS11*CS22*XXC2*XXS2 + SS22*SC12^2*XXC1^2
            + CC11*SC12^2*XXS2^2 - 2*CC11*SC12*SS12*XXC2*XXS2
            + 2*CC11*SS22*SC12*XXC2*XXS1 + CC22*SS12^2*XXC1^2
            + CC11*SS12^2*XXC2^2 + 2*CC11*CC22*SS12*XXS1*XXS2
            - CC22*SS11*SS22*XXC1^2 - CC11*SS11*SS22*XXC2^2
            - CC11*CC22*SS22*XXS1^2 - CC11*CC22*SS11*XXS2^2
```

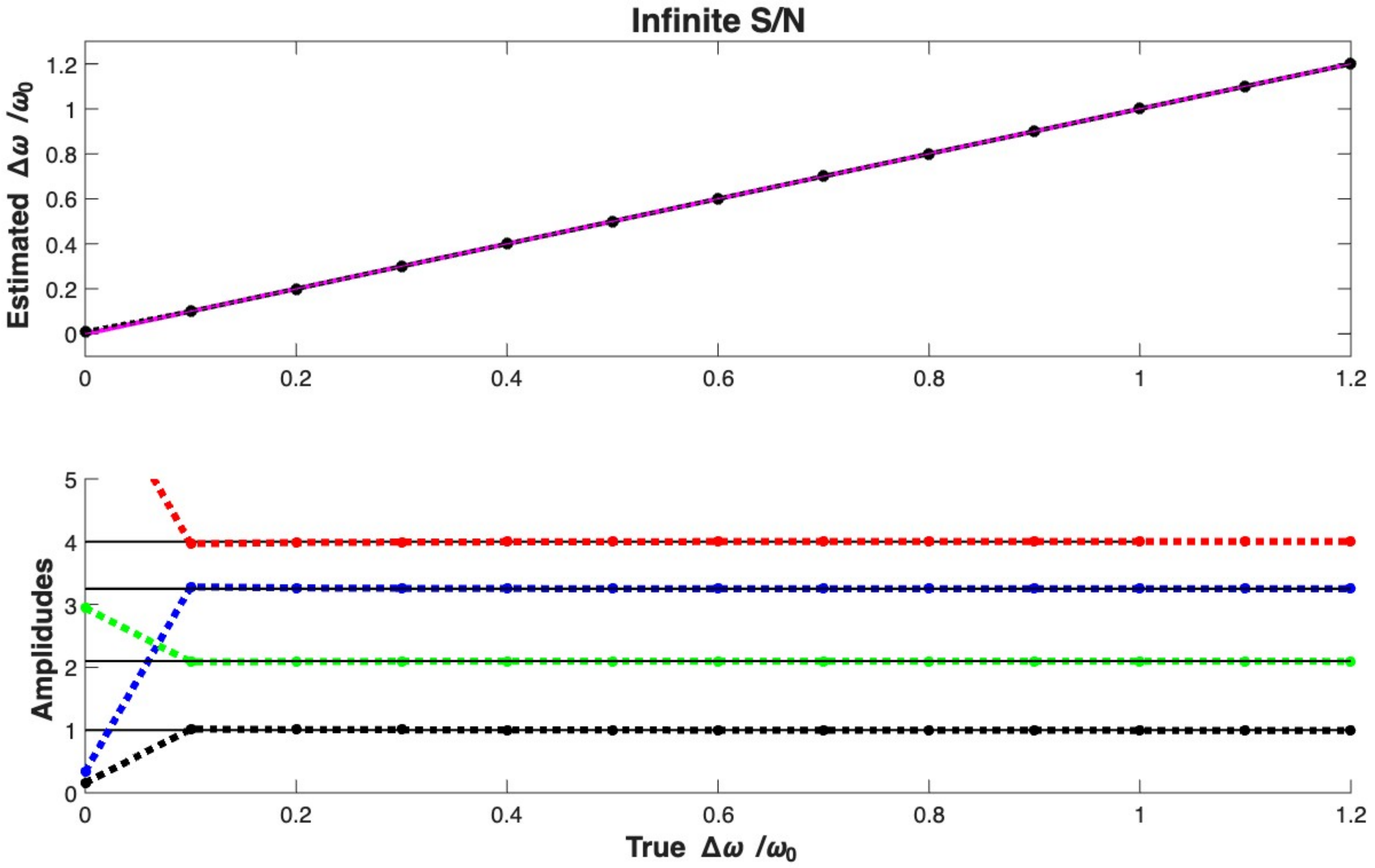
```
Bayes Denominator = ...
     - CC12^2*SS12^2 + SS11*SS22*CC12^2 + 2*CC12*CS11*CS22*SS12
    - 2*SS22*CC12*CS11*SC12 - 2*SS11*CC12*CS12*CS22
     + 2*CC12*CS12*SC12*SS12 - CS11^2*CS22^2 + CC22*SS22*CS11^2
     + 2*CS11*CS12*CS22*SC12 - 2*CC22*CS11*CS12*SS12
     - CS12^2*SC12^2 + CC22*SS11*CS12^2 + CC11*SS11*CS22^2
     - 2*CC11*CS22*SC12*SS12 + CC11*SS22*SC12^2
     + CC11*CC22*SS12^2 - CC11*CC22*SS11*SS22
```

Double Bayesian Periodogram

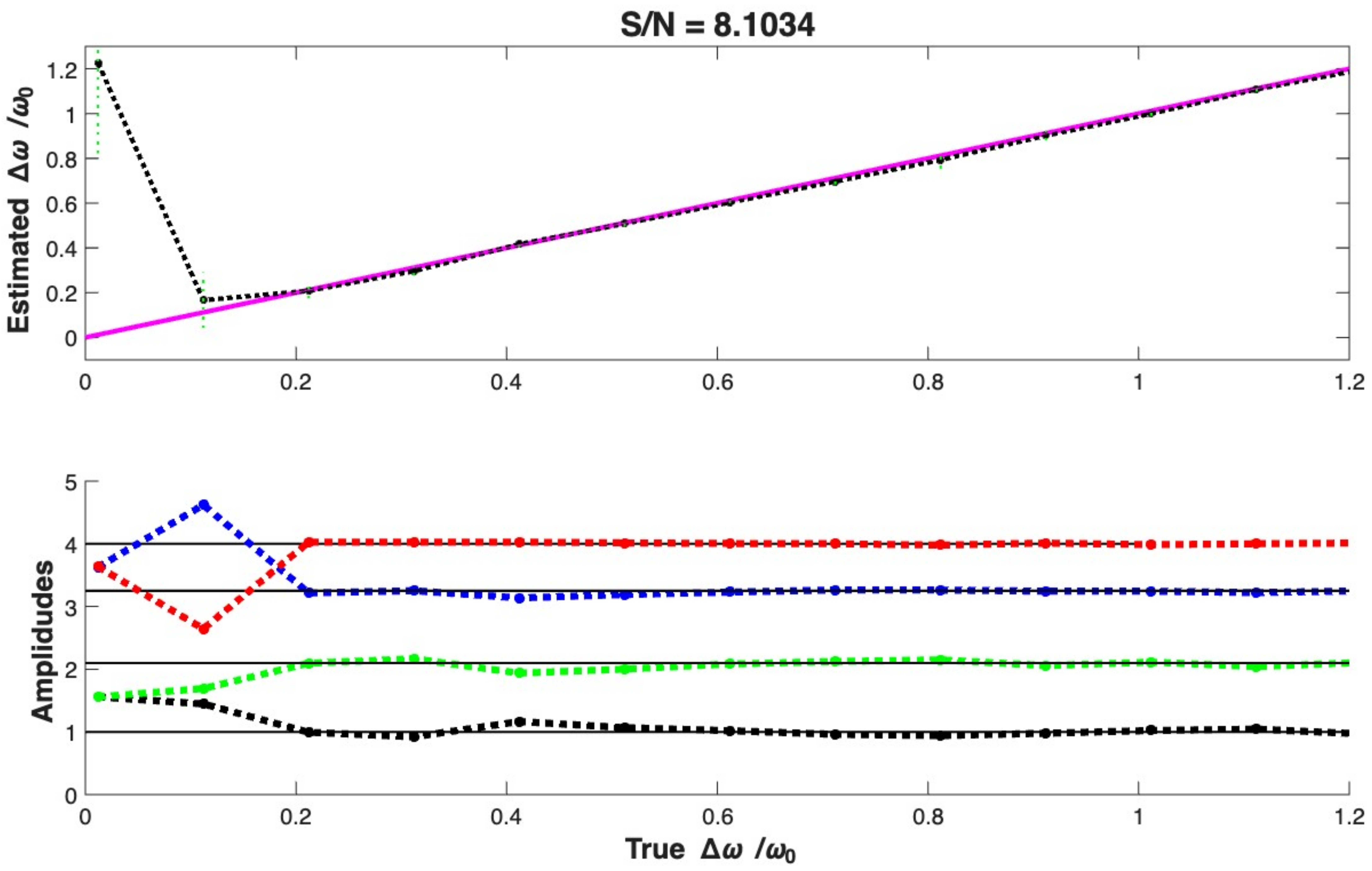
= Bayes Numerator / Bayes Denominator

Two Sinusoids (simulated)

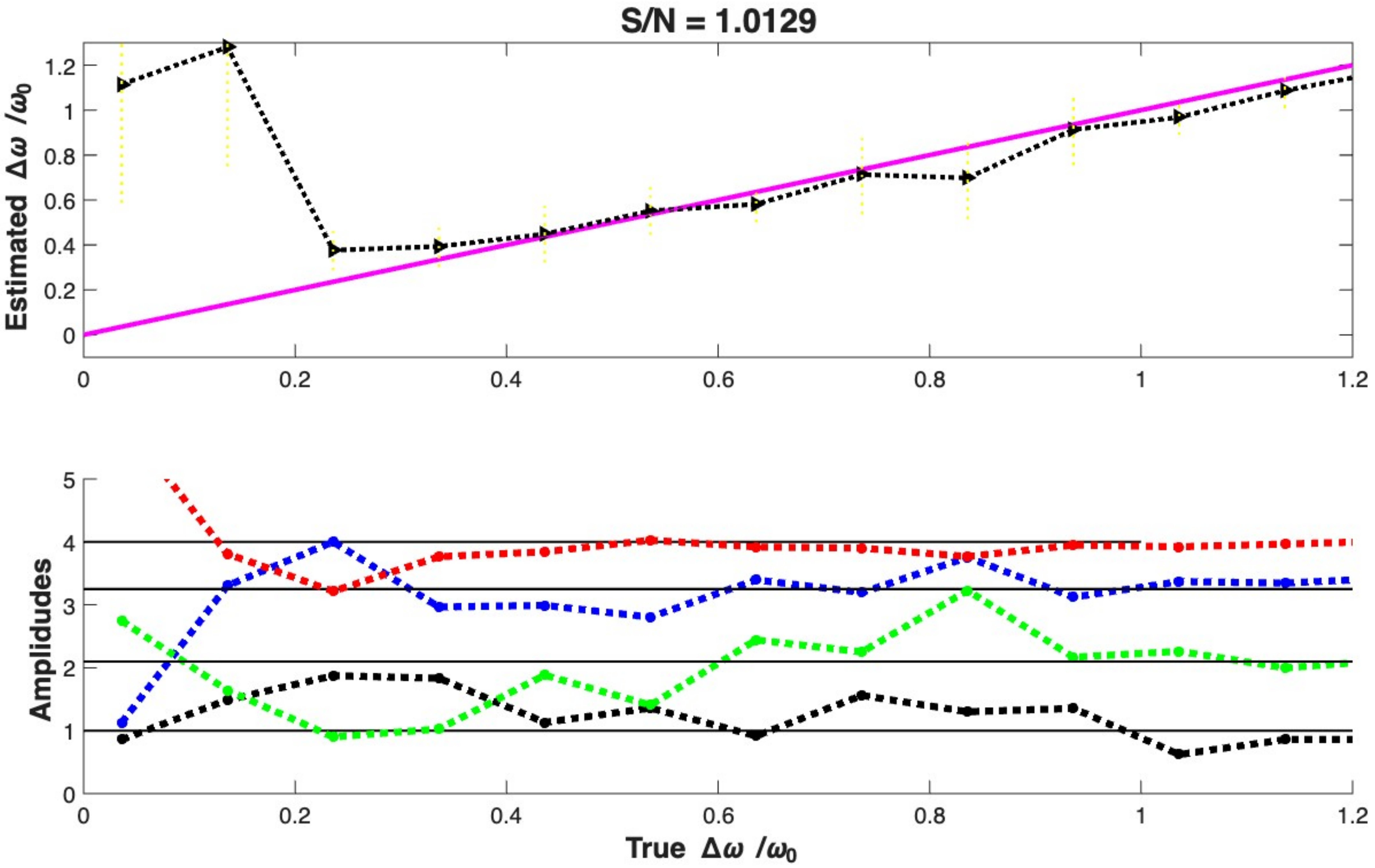




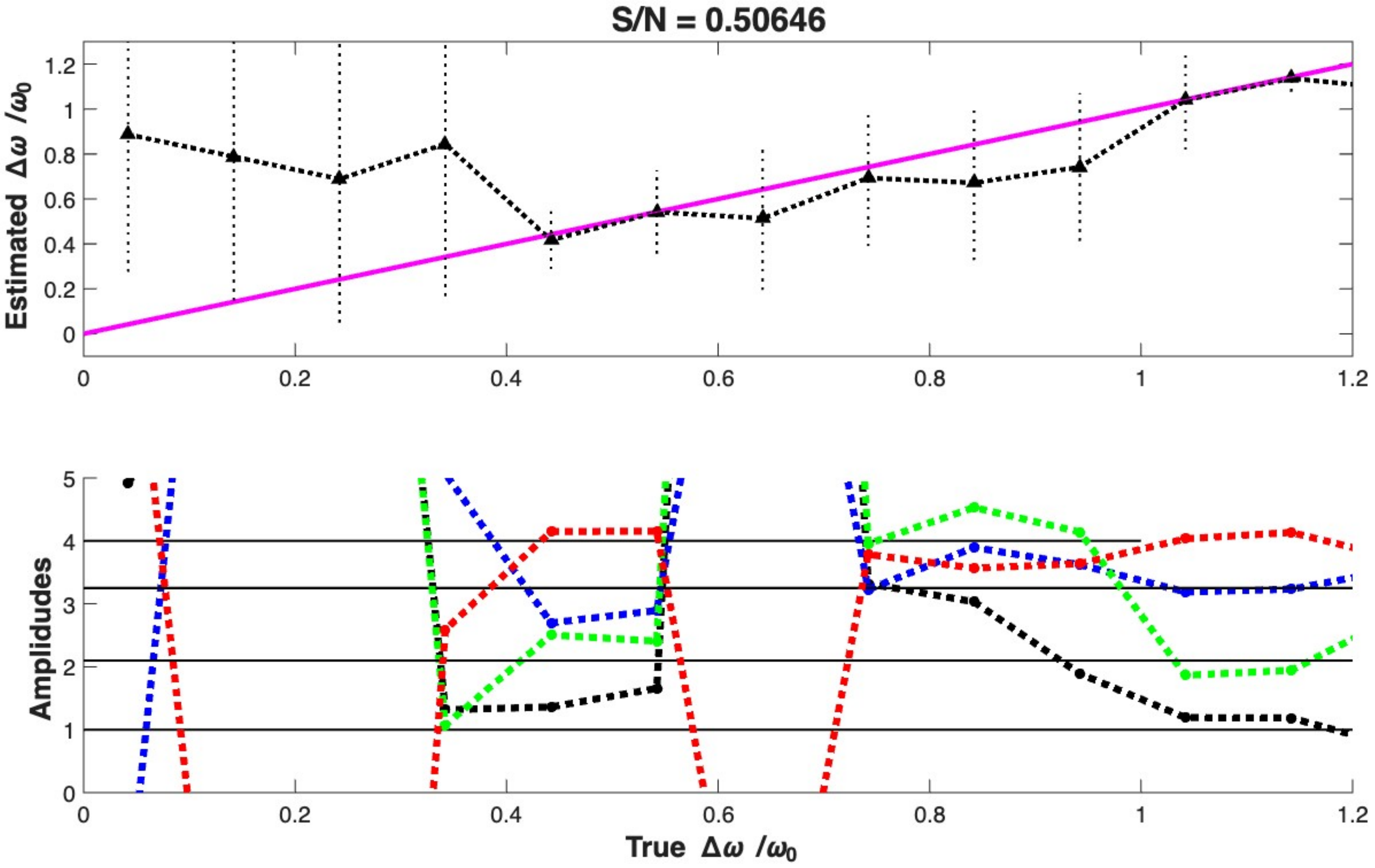
Infinite S/N
Estimated $\Delta\omega\ /\omega_0$
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
Amplidudes
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1
1.2
True $\Delta\omega\ /\omega_0$



S/N = 8.1034
Estimated $\Delta\omega\ /\omega_0$
Amplitudes
True $\Delta\omega\ /\omega_0$
0
0.2
0.4
0.6
0.8
1
1.2
1
2
3
4
5

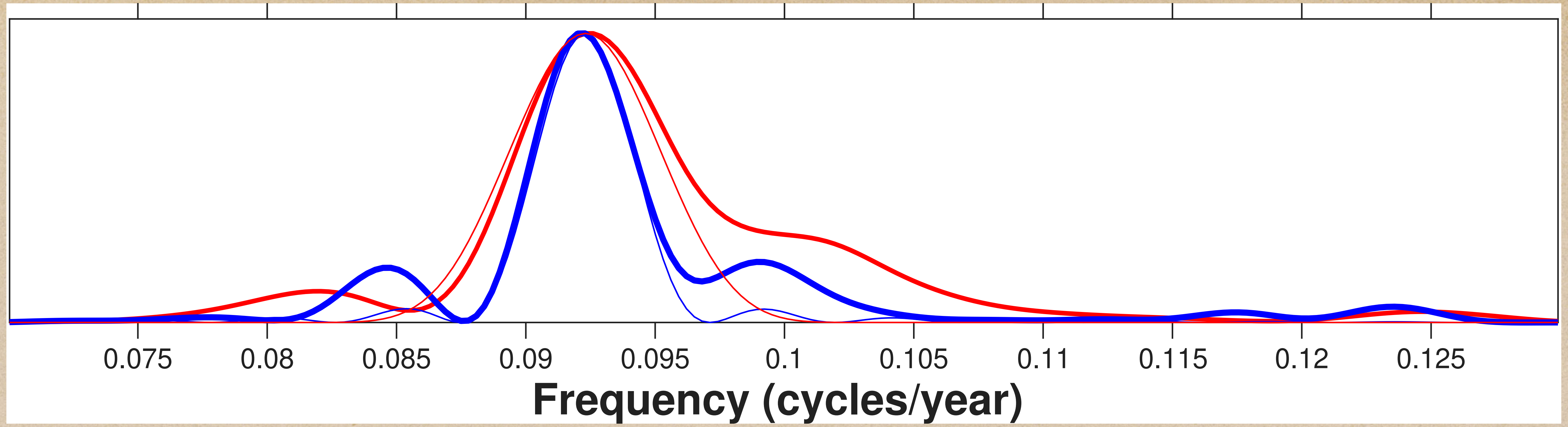


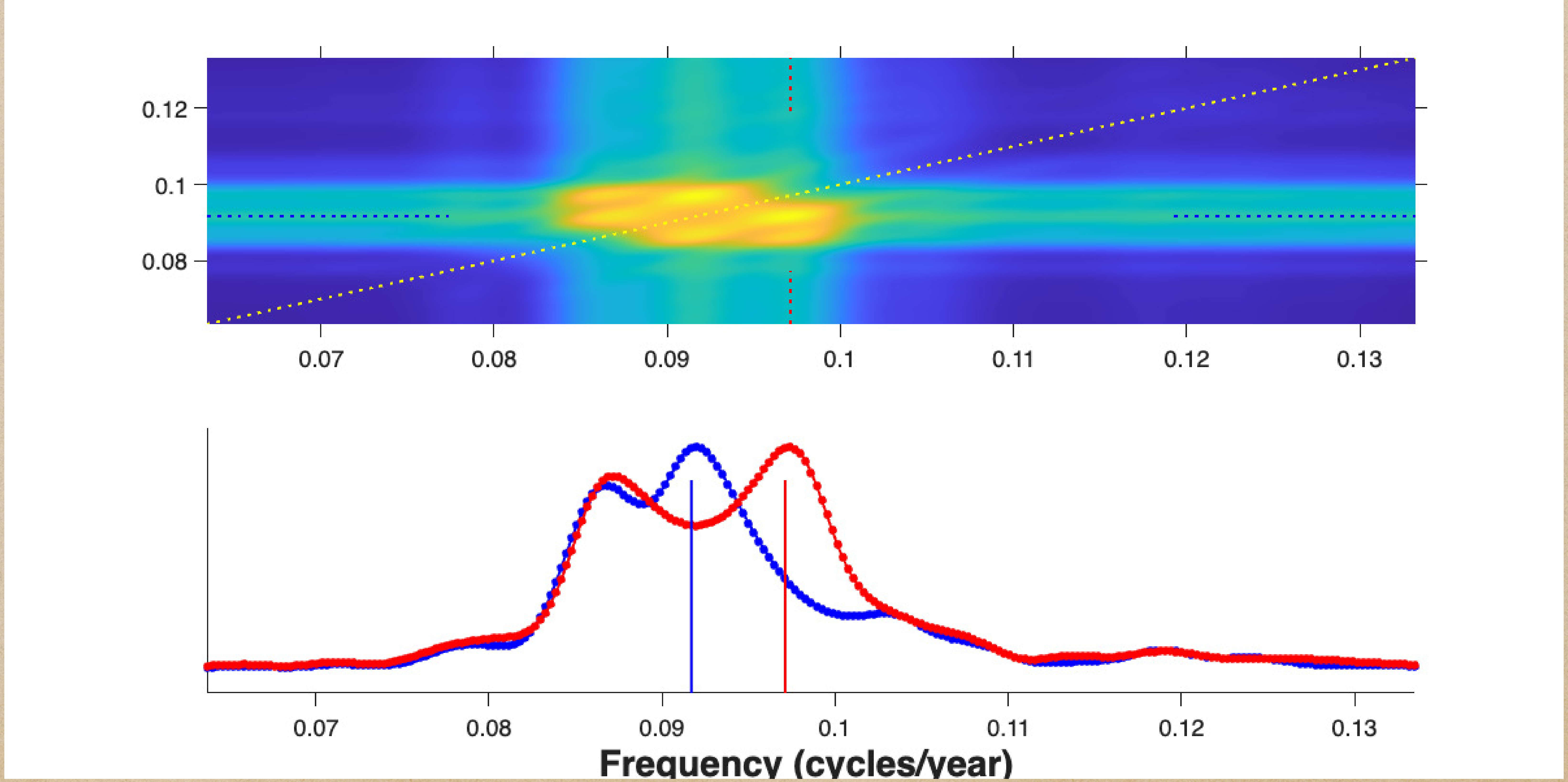
S/N = 1.0129
Estimated Δω /ω0
Amplidudes
True Δω /ω0
0
0.2
0.4
0.6
0.8
1
1.2
1
2
3
4
5



S/N = 0.50646
Estimated $\Delta\omega$ $/\omega_0$
0
0.2
0.4
0.6
0.8
1
1.2
0
0.2
0.4
0.6
0.8
1
1.2
Amplidudes
0
1
2
3
4
5
0
0.2
0.4
0.6
0.8
1
1.2
True $\Delta\omega$ $/\omega_0$

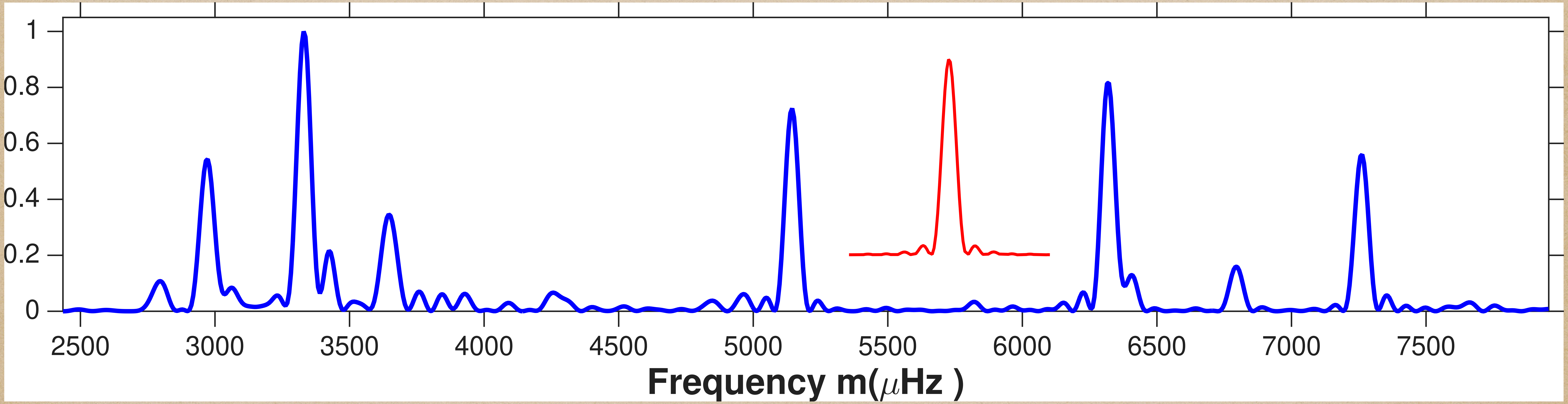
Sunspot Number Time Series

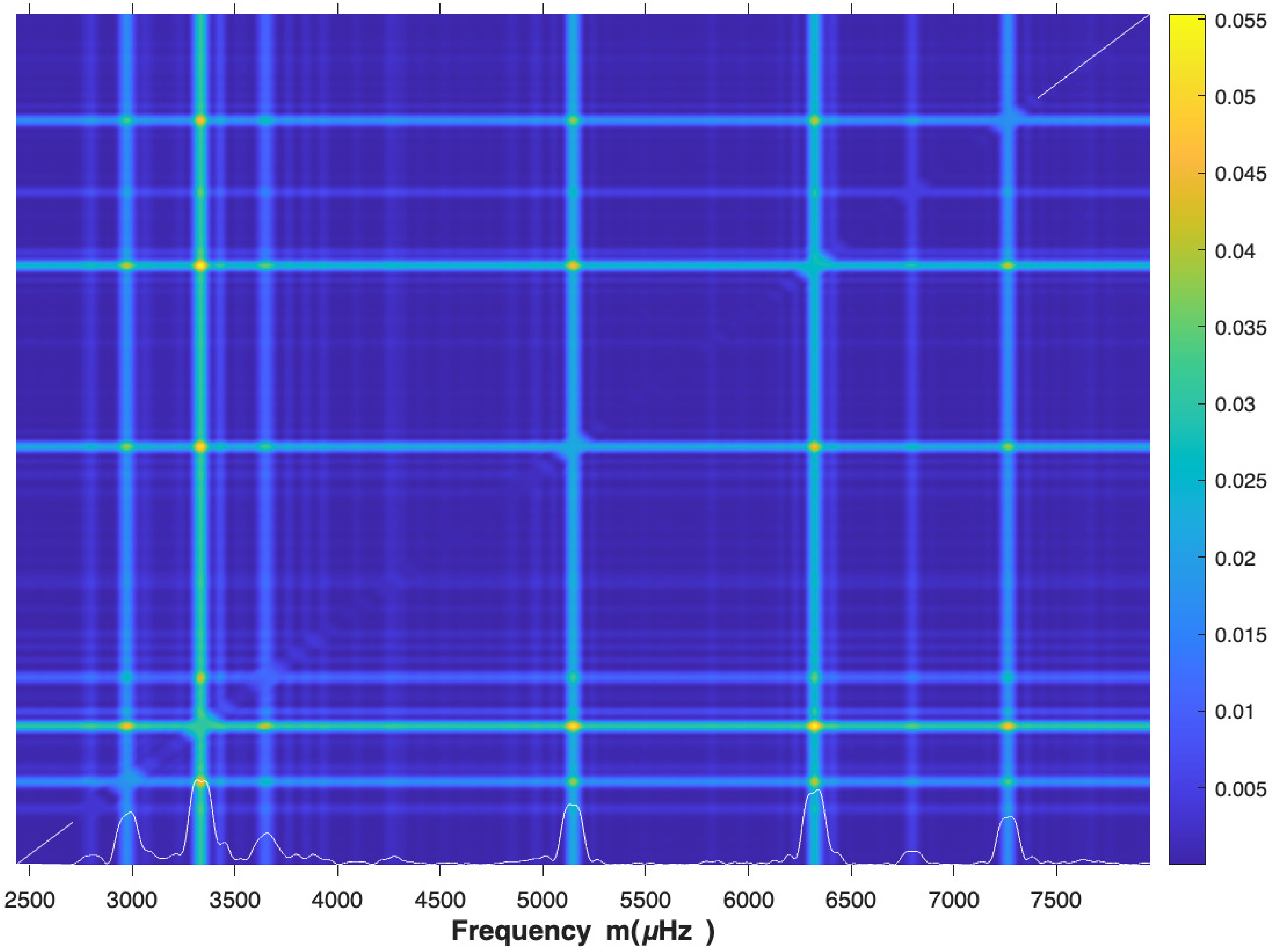




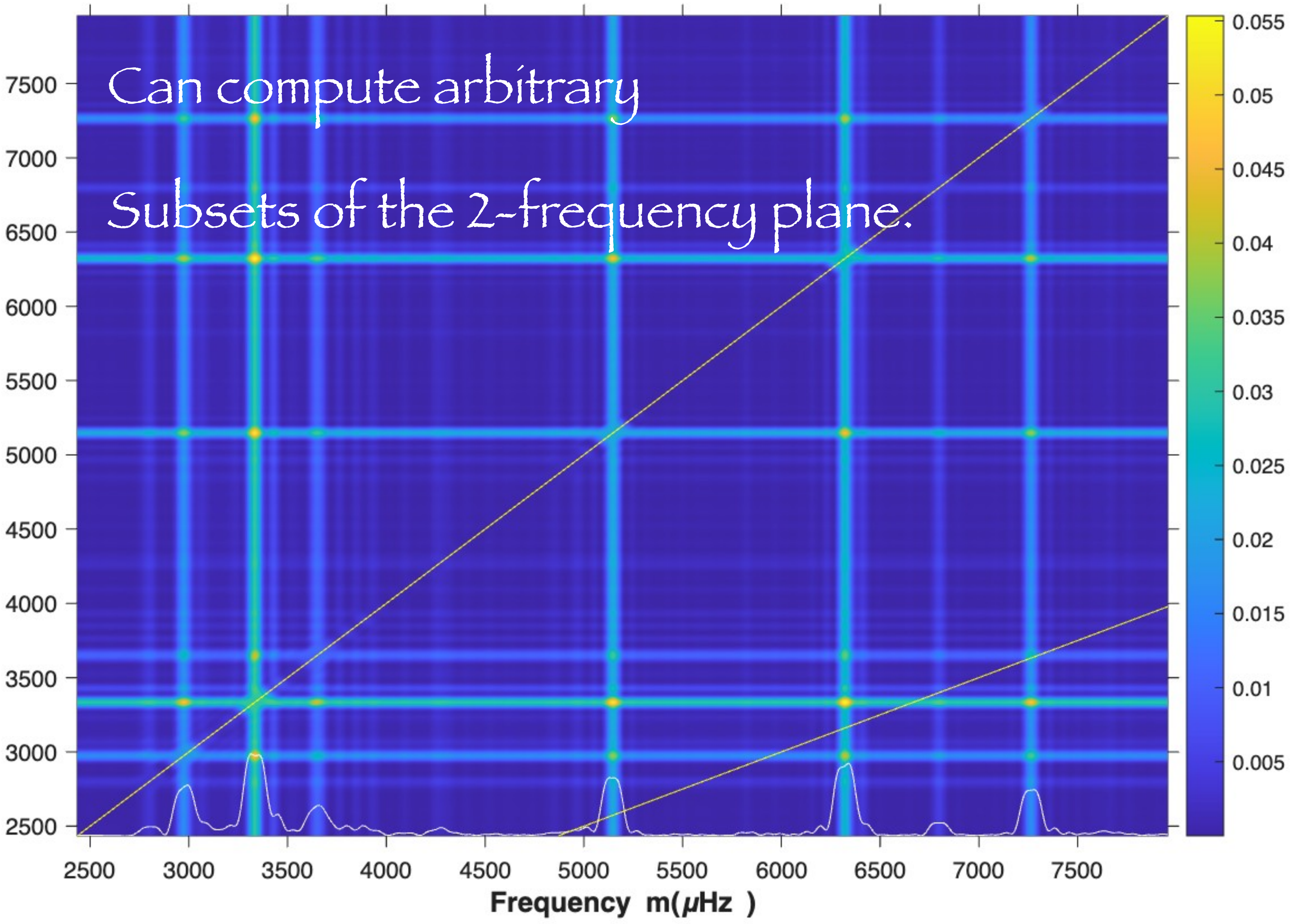
The Pulsating White Dwarf WD J0135+5722

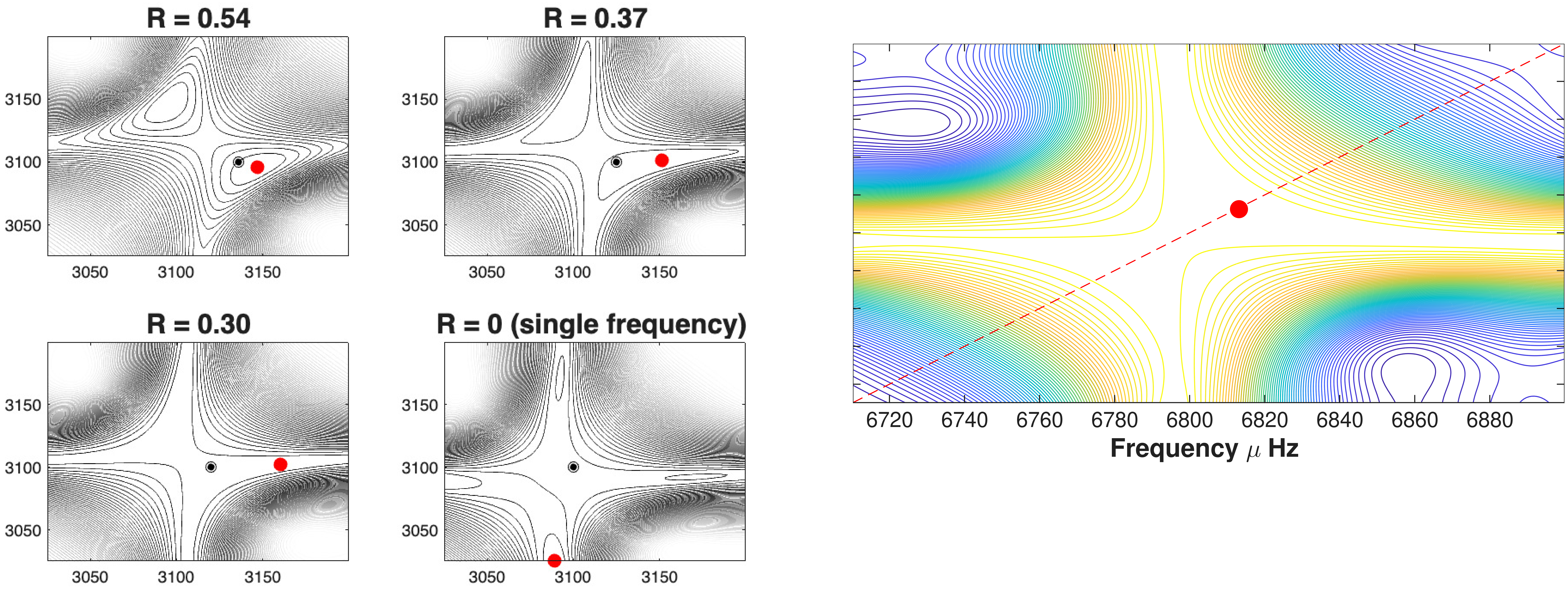
De Geronimo, Uzundag, Rebassa-Mansergas, Brown, Kilic, Corsico, Jewett and Moss
2025, arXiv:2501.1366





0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
2500
3000
3500
4000
4500
5000
5500
6000
6500
7000
7500
Frequency m(μHz)


Can compute arbitrary
Subsets of the 2-frequency plane.
7500
7000
6500
6000
5500
5000
4500
4000
3500
3000
2500
2500
3000
3500
4000
4500
5000
5500
6000
6500
7000
7500
Frequency m(μHz)
0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005



R = 0.54
R = 0.37
R = 0.30
R = 0 (single frequency)
3050
3100
3150
6720
6740
6760
6780
6800
6820
6840
6860
6880
Frequency μ Hz

Future Work and Open Questions

Applications? Asteroseismology, pulsating stars, …

Fix the singularities at $\omega_1 = \omega_2$

Uncertainties (posteriors for the amplitudes)

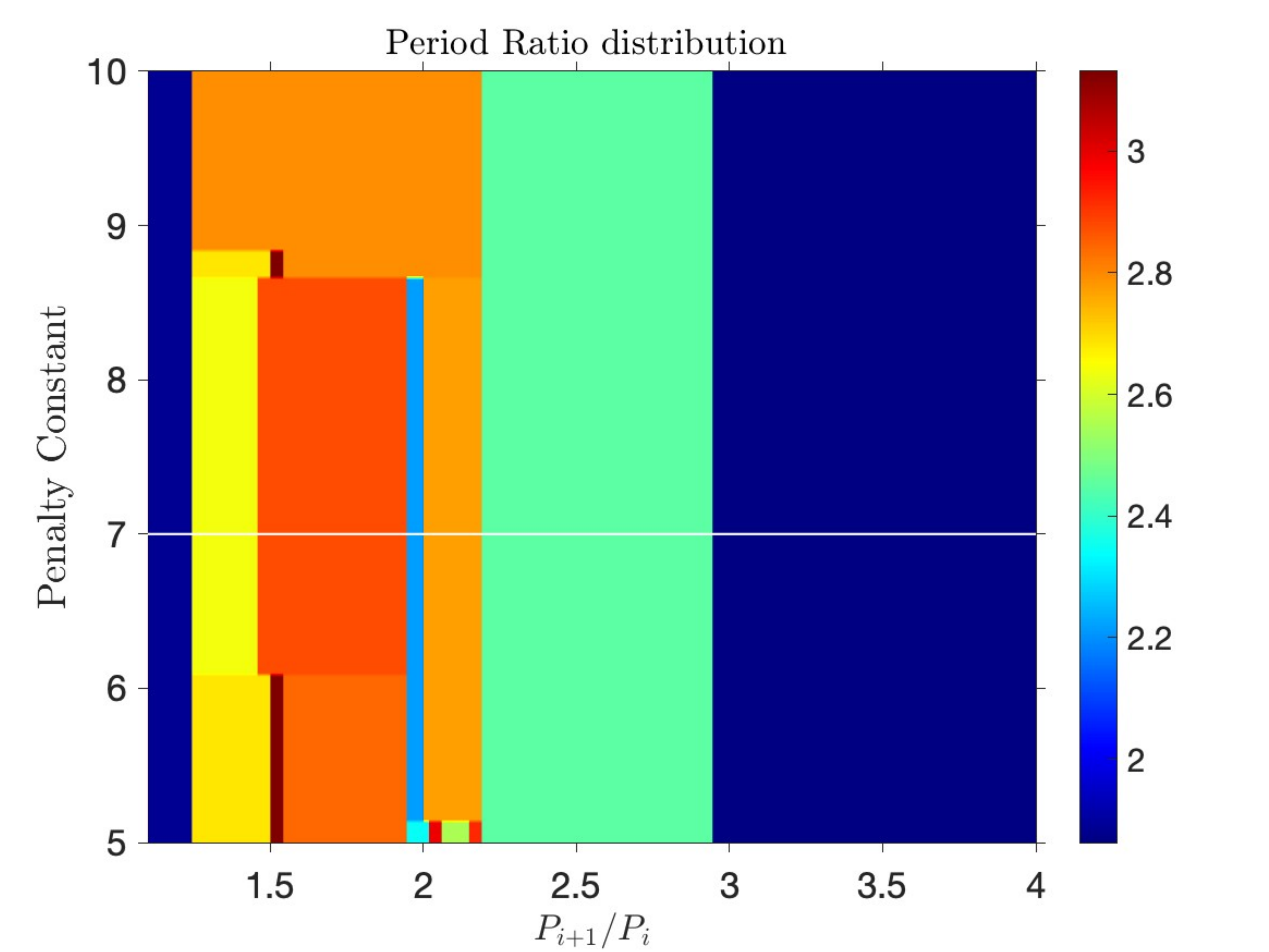
Component amplitudes from Bayesian Omnigrams

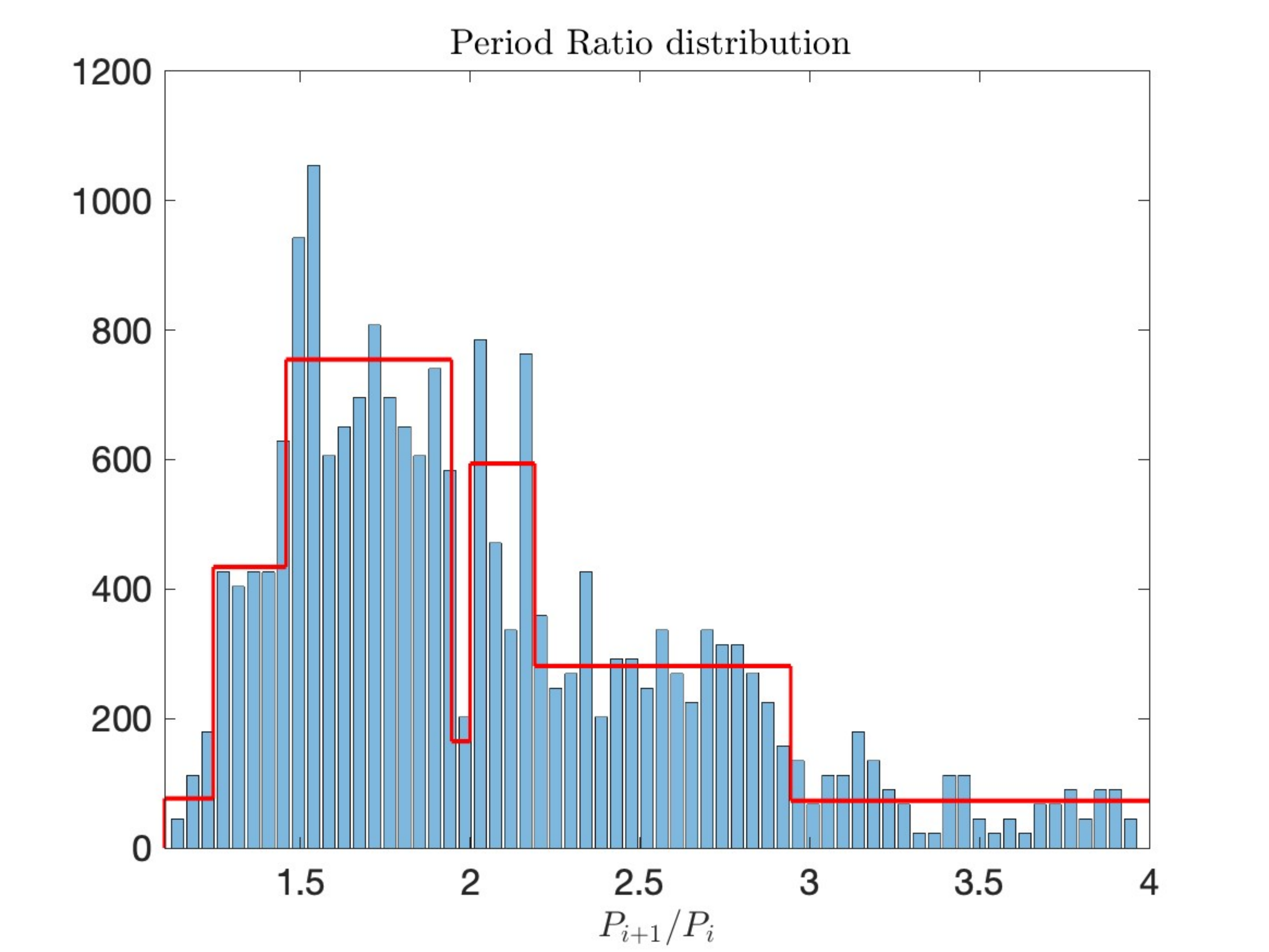
Is there a multifrequency Fourier Transform: Omnigram = $|FT|^2$?

Omnigram for a frequency continuum?

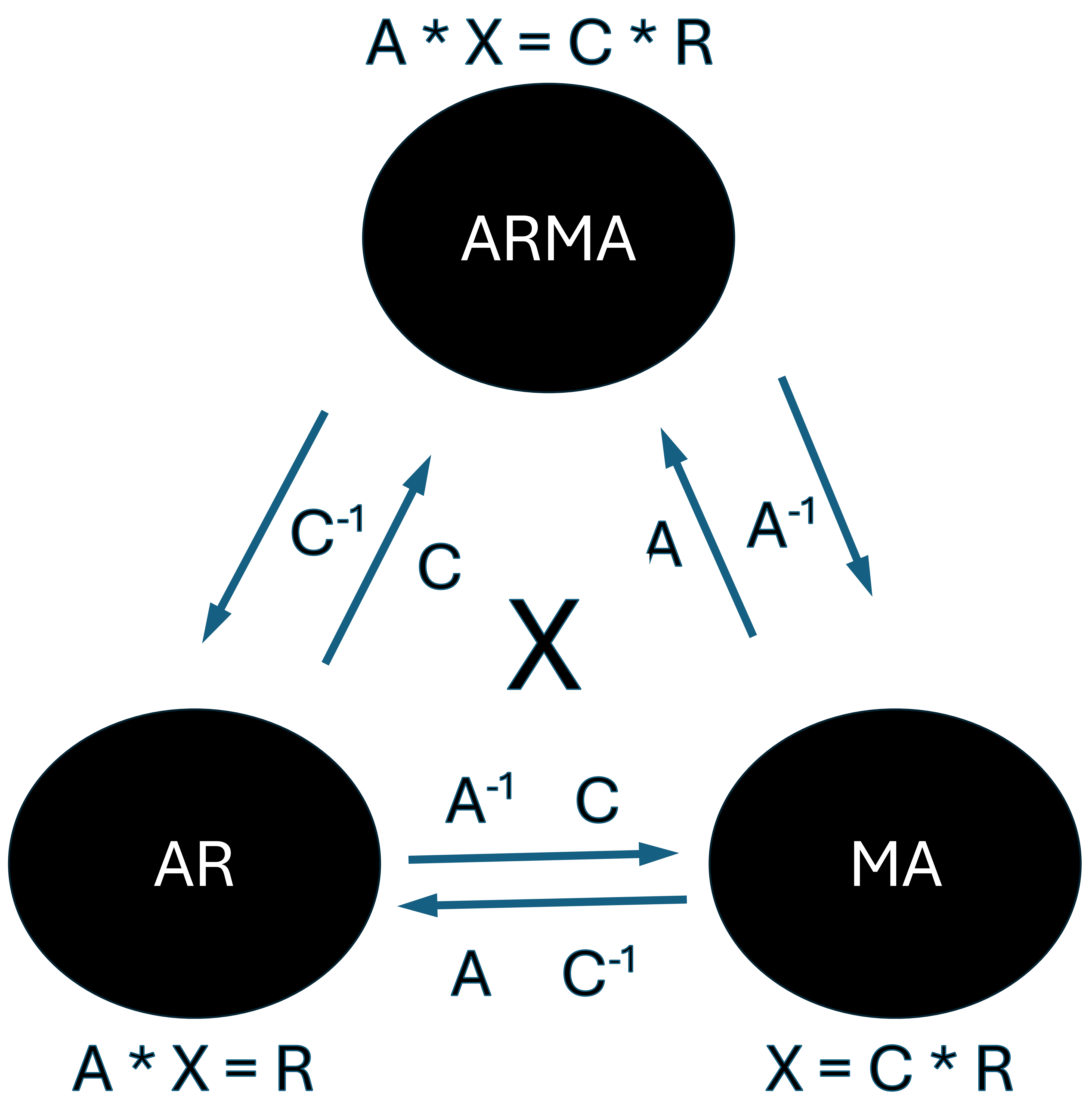
Bayesian Block histograms accounting for measurement error.
Exact blind deconvolution for generalized ARMA models

Bayesian Block Histograms: Empty Blocks?

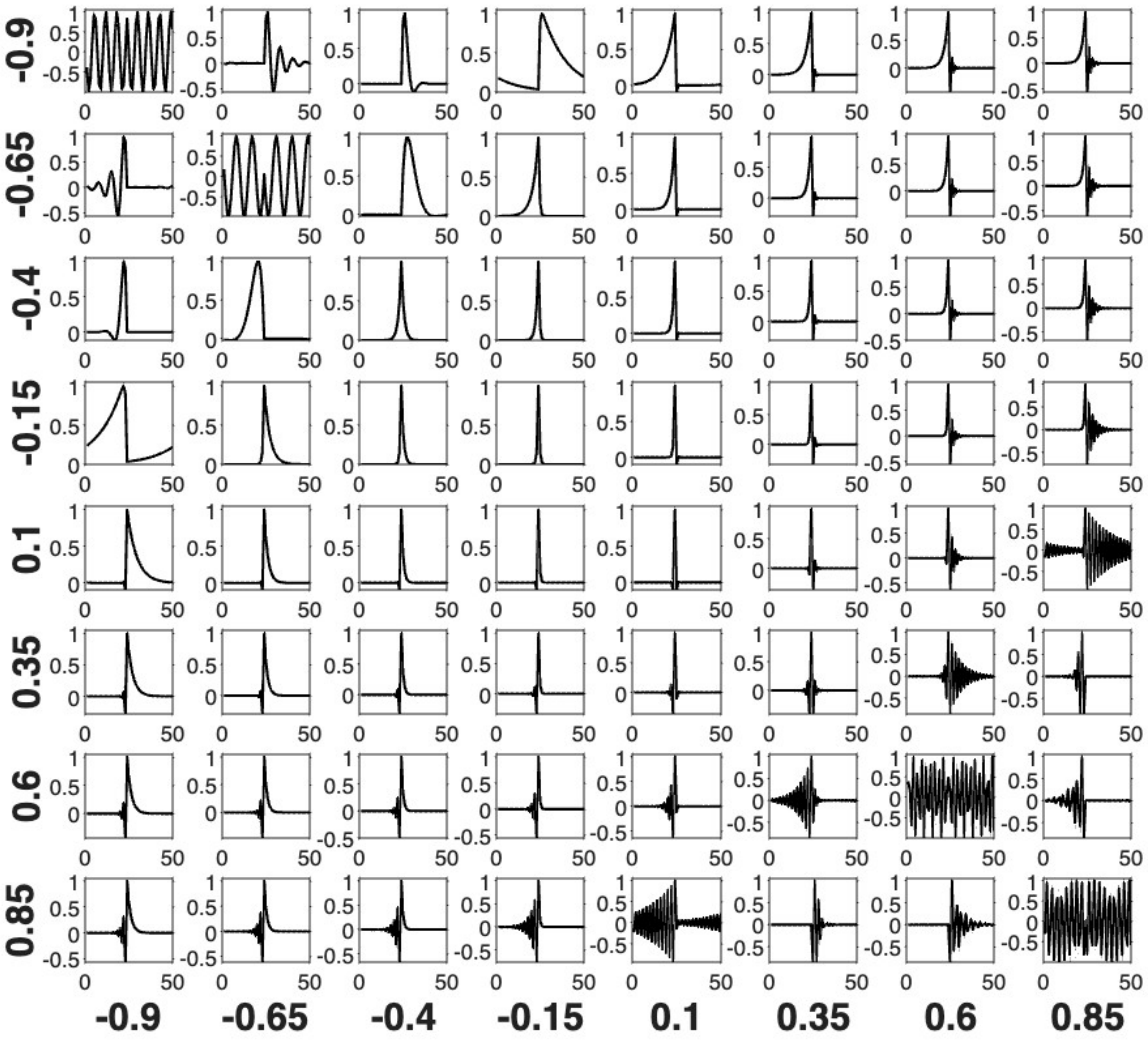


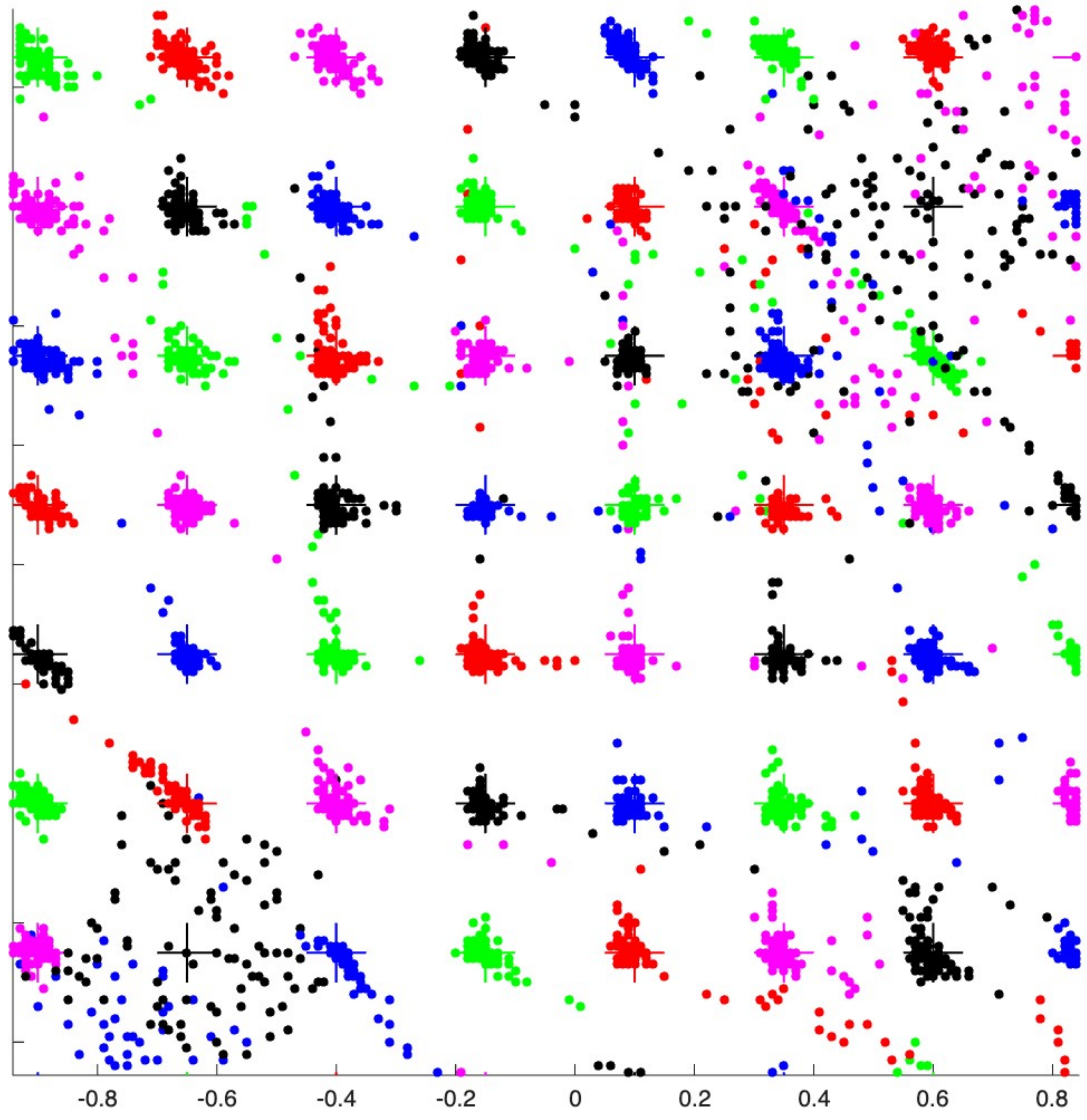


The ARMA Convolution Group

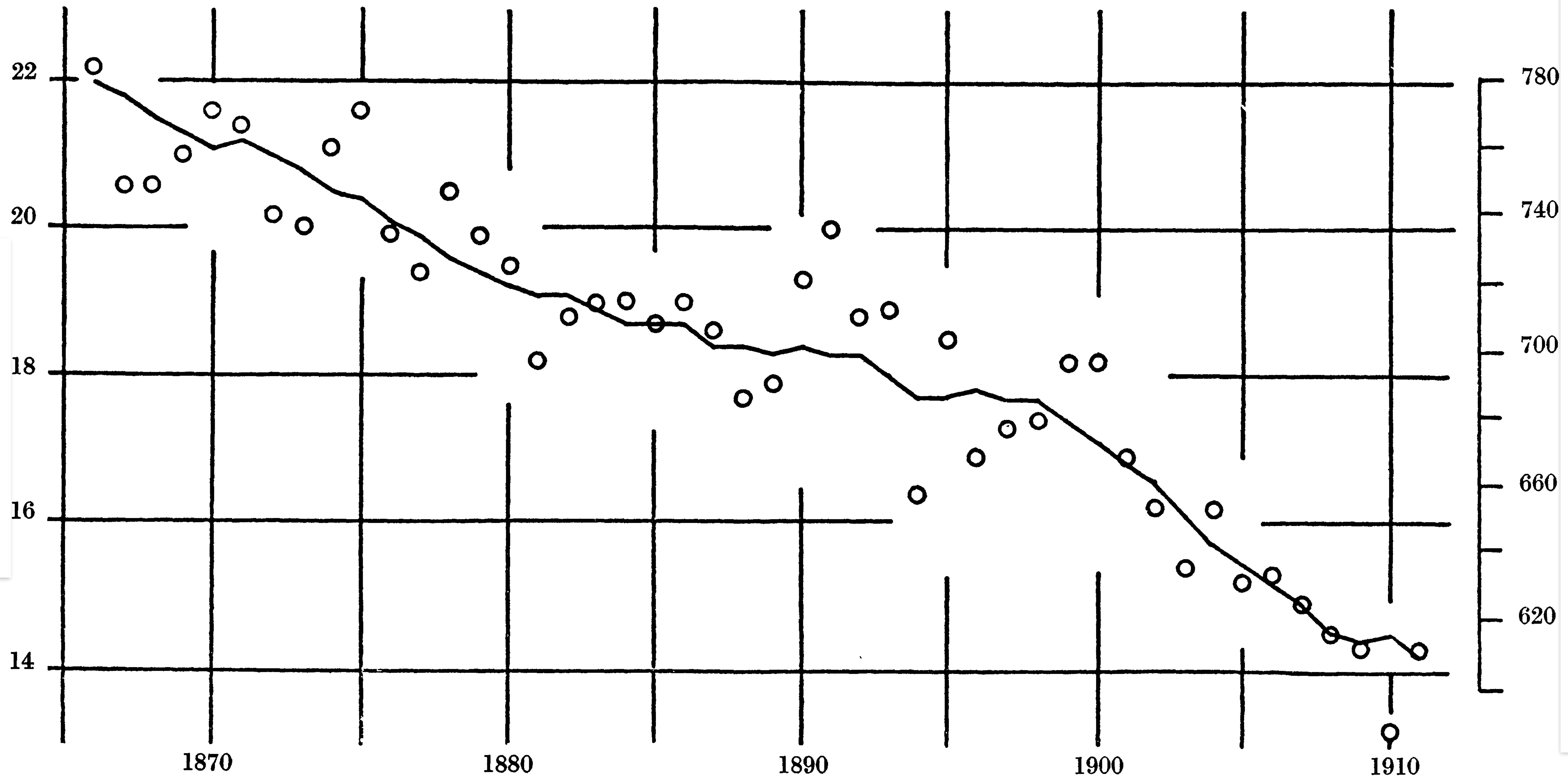


Blind Deconvolution of acausal/non-minimum-delay autoregressive processes.





G. Udny Yule
Journal of the Royal Statistical Society, Vol. 89, (1926), pp. 1-63.



B 2

G. Udny Yule

Journal of the Royal Statistical Society, Vol. 89, (1926), pp. 1-63.

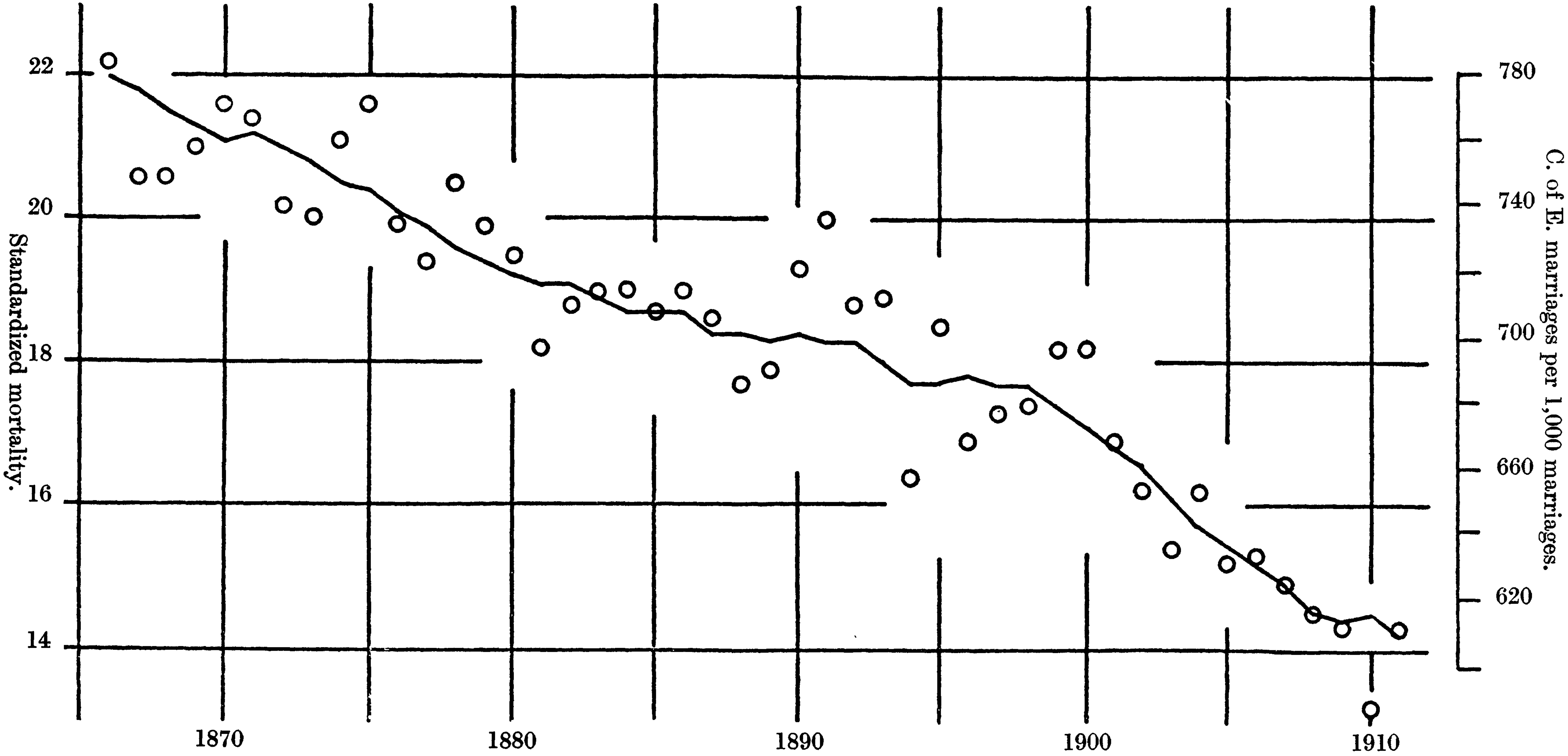


Fig. 1.—Correlation between standardized mortality per 1,000 persons in England and Wales (circles), and the proportion of Church of England marriages per 1,000 of all marriages (line), 1866–1911. $r = +0{\cdot}9512$.

Why do we Sometimes get Nonsense-Correlations between Time-Series?
--A Study in Sampling and the Nature of Time-Series
G. Udny Yule
Journal of the Royal Statistical Society, Vol. 89, (1926), pp. 1-63.

Now I suppose it is possible, given a little ingenuity and goodwill, to rationalize very nearly anything. And I can imagine some enthusiast arguing that the fall in the proportion of Church of England marriages is simply due to the Spread of Scientific Thinking since 1866, and the fall in mortality is also clearly to be ascribed to the Progress of Science; hence both variables are largely or mainly influenced by a common factor and consequently ought to be highly correlated. But most people would, I think, agree with me that the correlation is simply sheer nonsense; that it has no meaning whatever; that it is absurd to suppose that the two variables in question are in any sort of way, however indirect, causally related to one another.