

 Tiana_Athriel

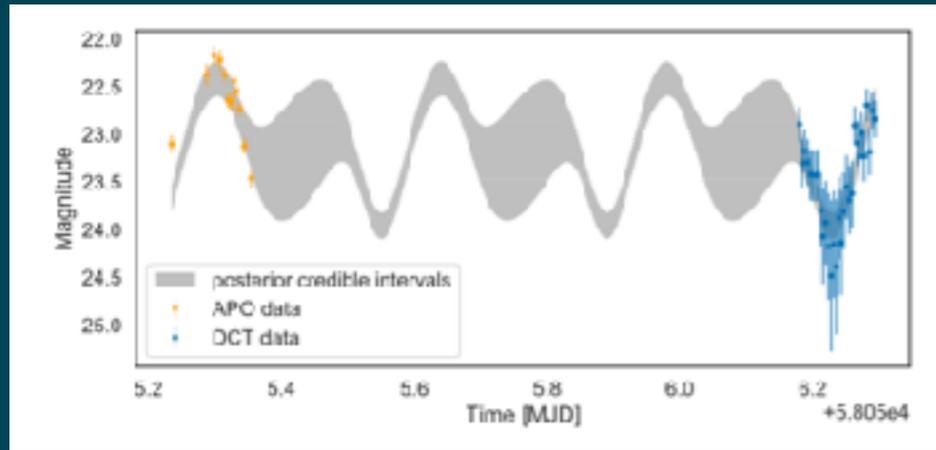
 dhuppenkothen

Fun Statistics with Fourier Spectra

Daniela Huppenkothen

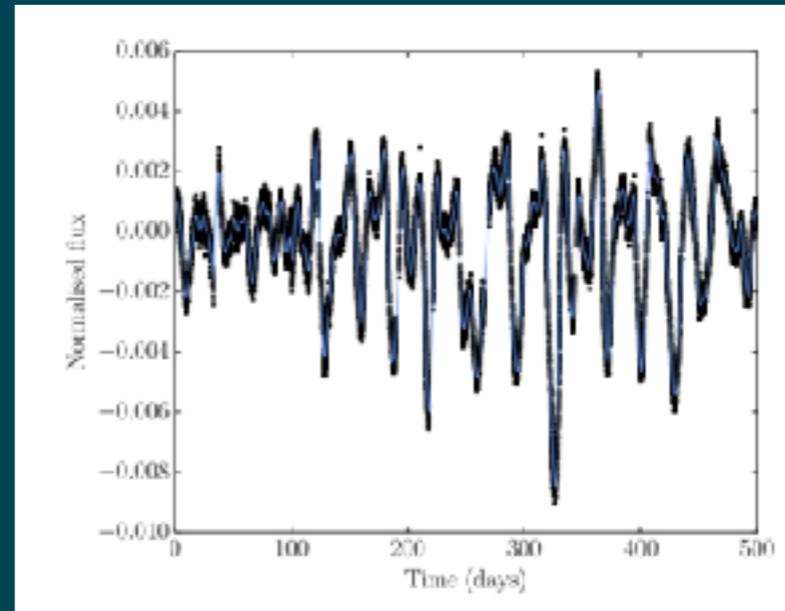
DIRAC Institute, UW Seattle

asteroids



Bolin et al (2017)

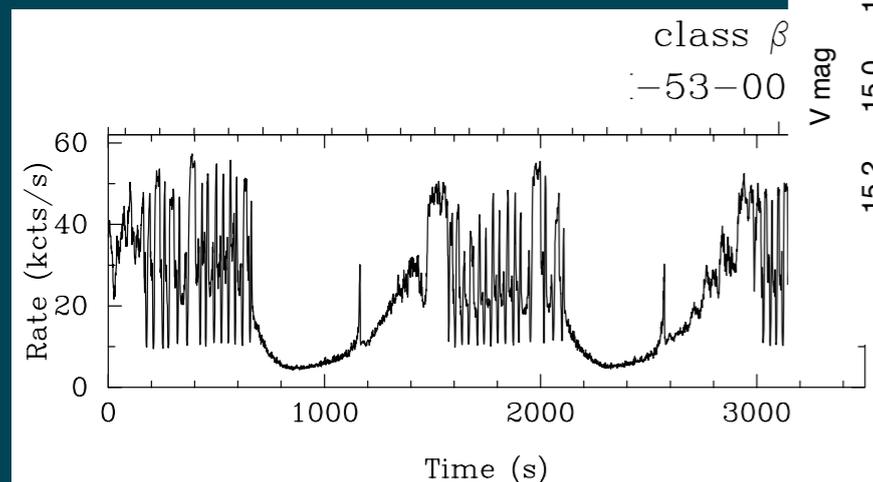
stars



Angus et al (2018)

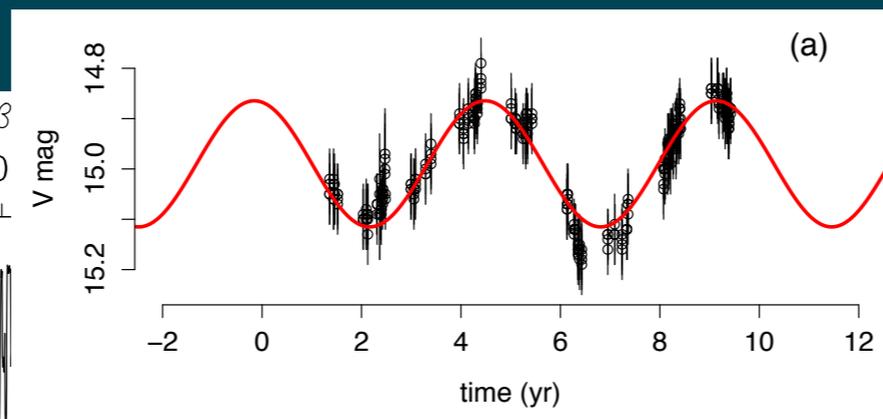
Almost all things in the universe are variable

black holes



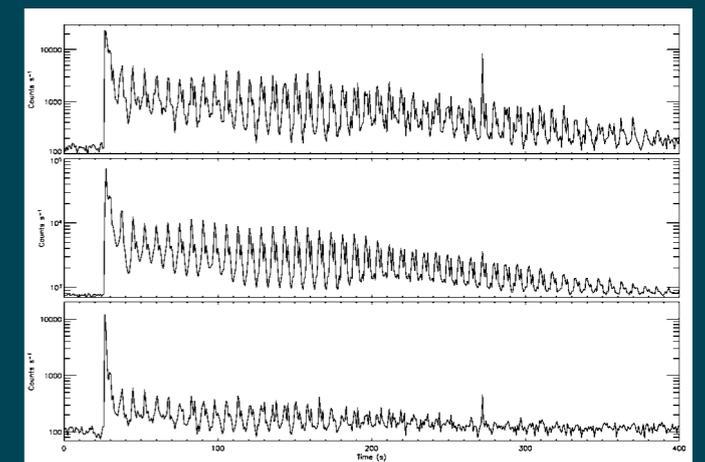
Belloni et al (2000)

Vaughan et al (2016)



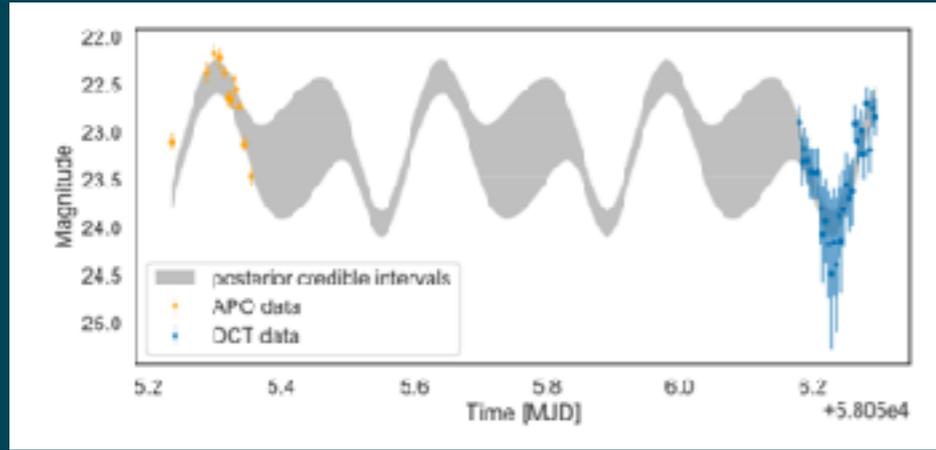
AGN

Strohmayer & Watts (2005)



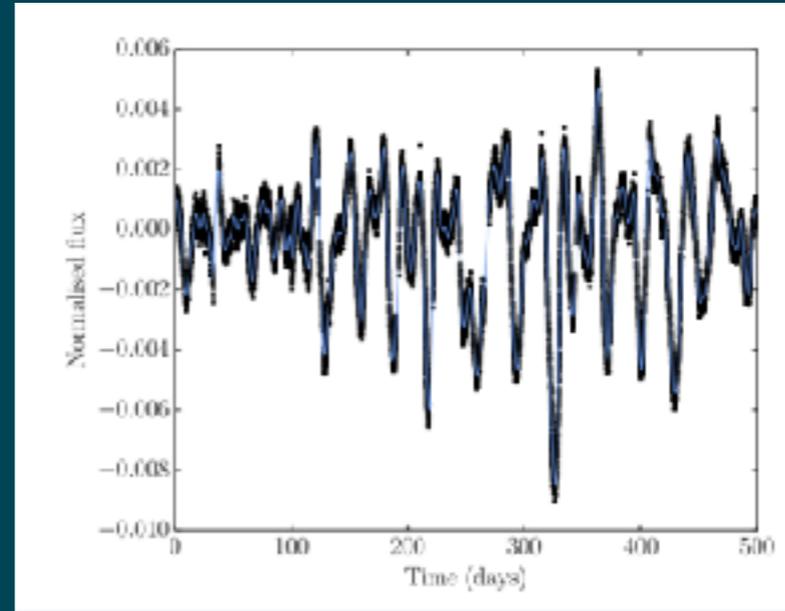
neutron stars

asteroids



Bolin et al (2017)

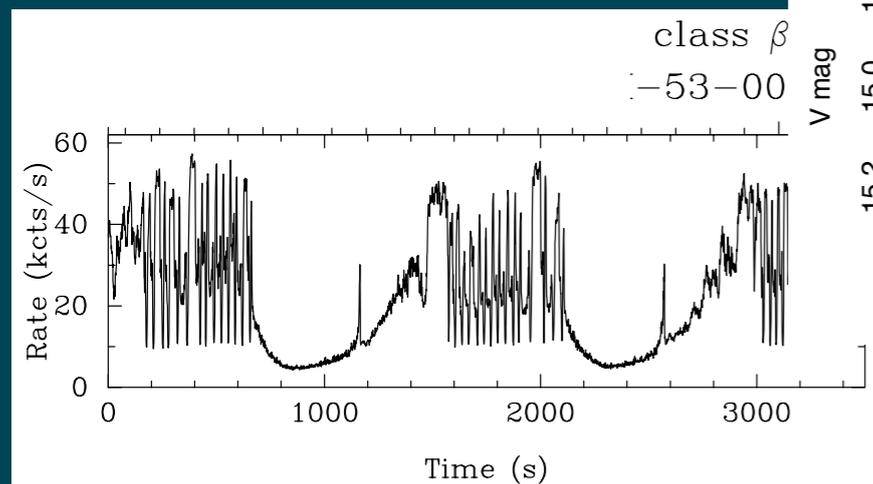
stars



Angus et al (2018)

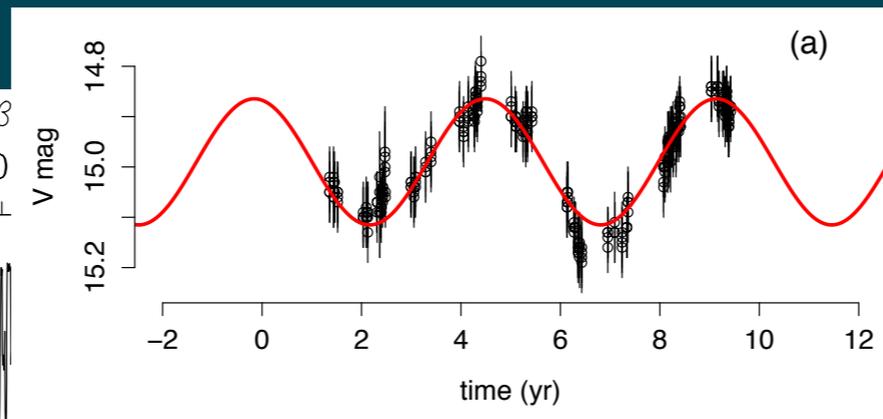
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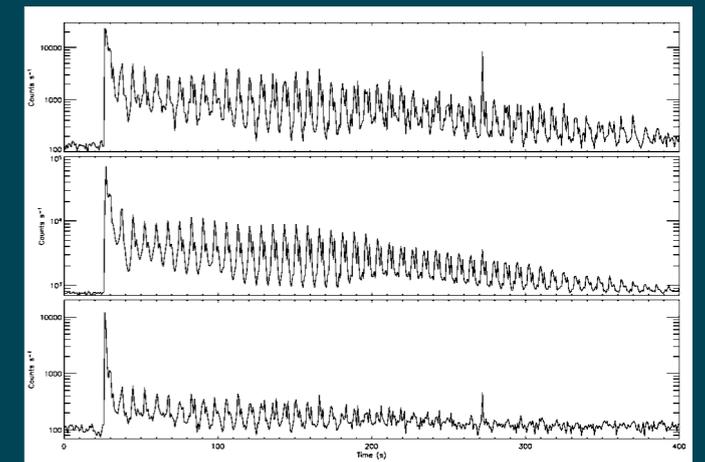
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neutron stars

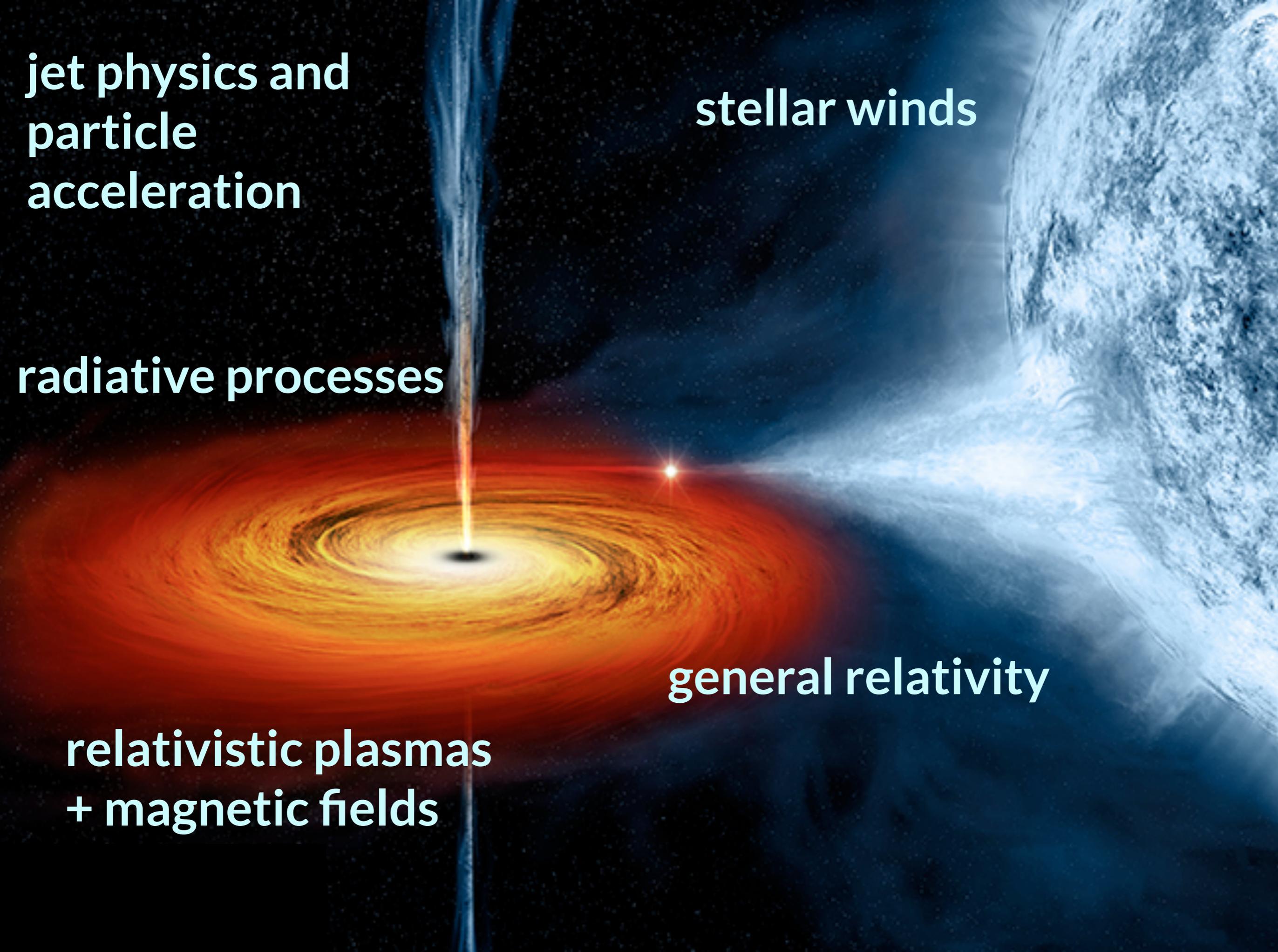
**jet physics and
particle
acceleration**

stellar winds

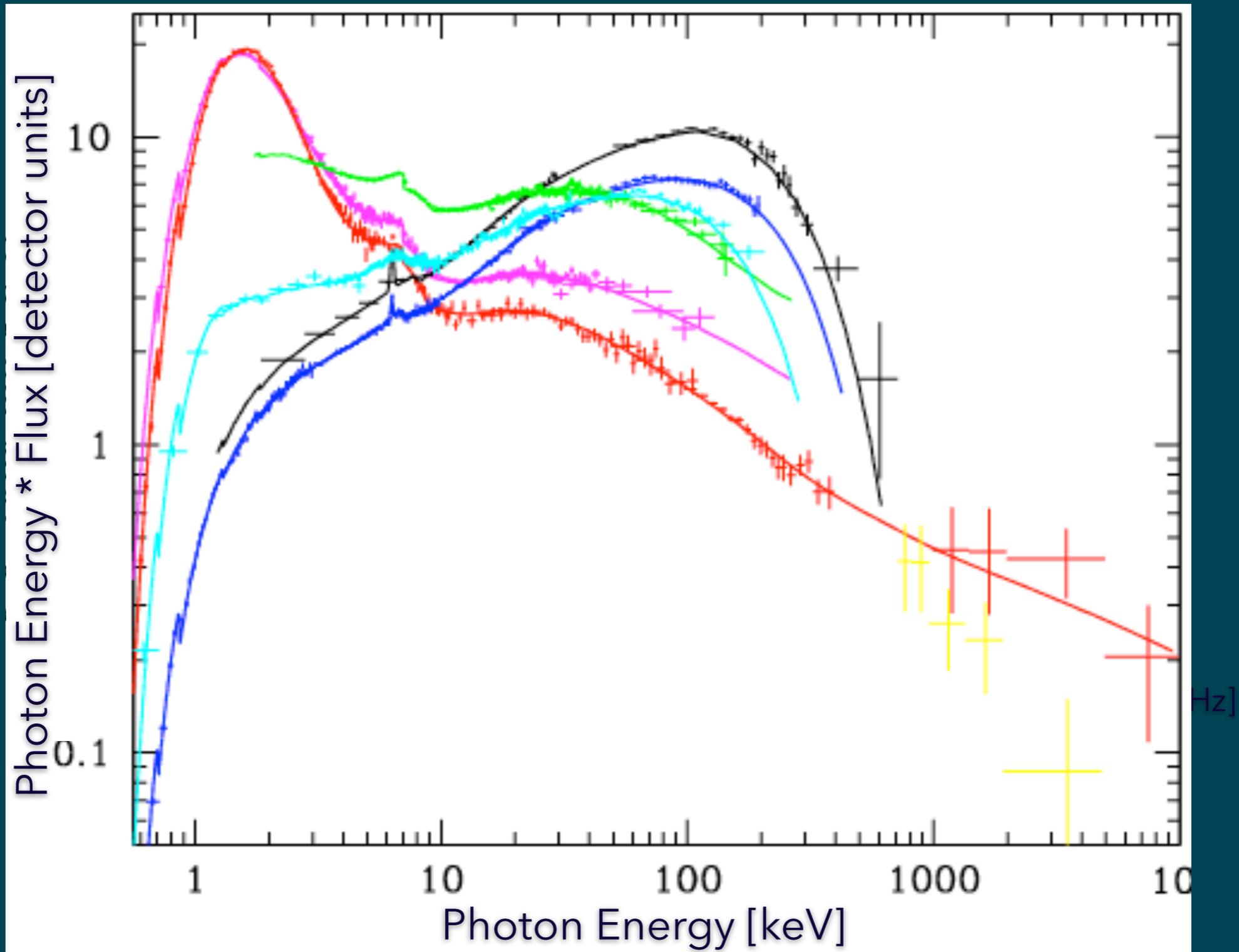
radiative processes

general relativity

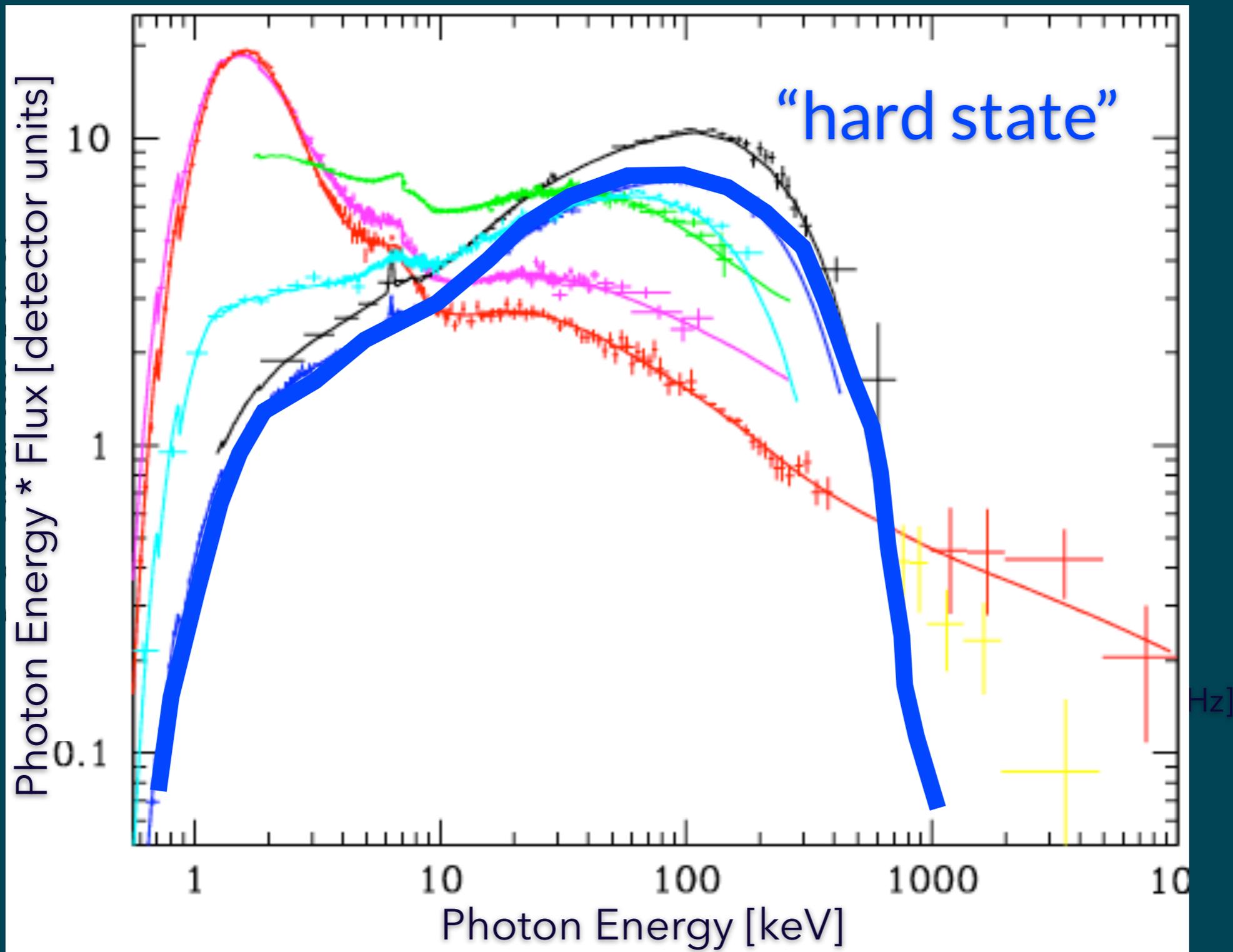
**relativistic plasmas
+ magnetic fields**



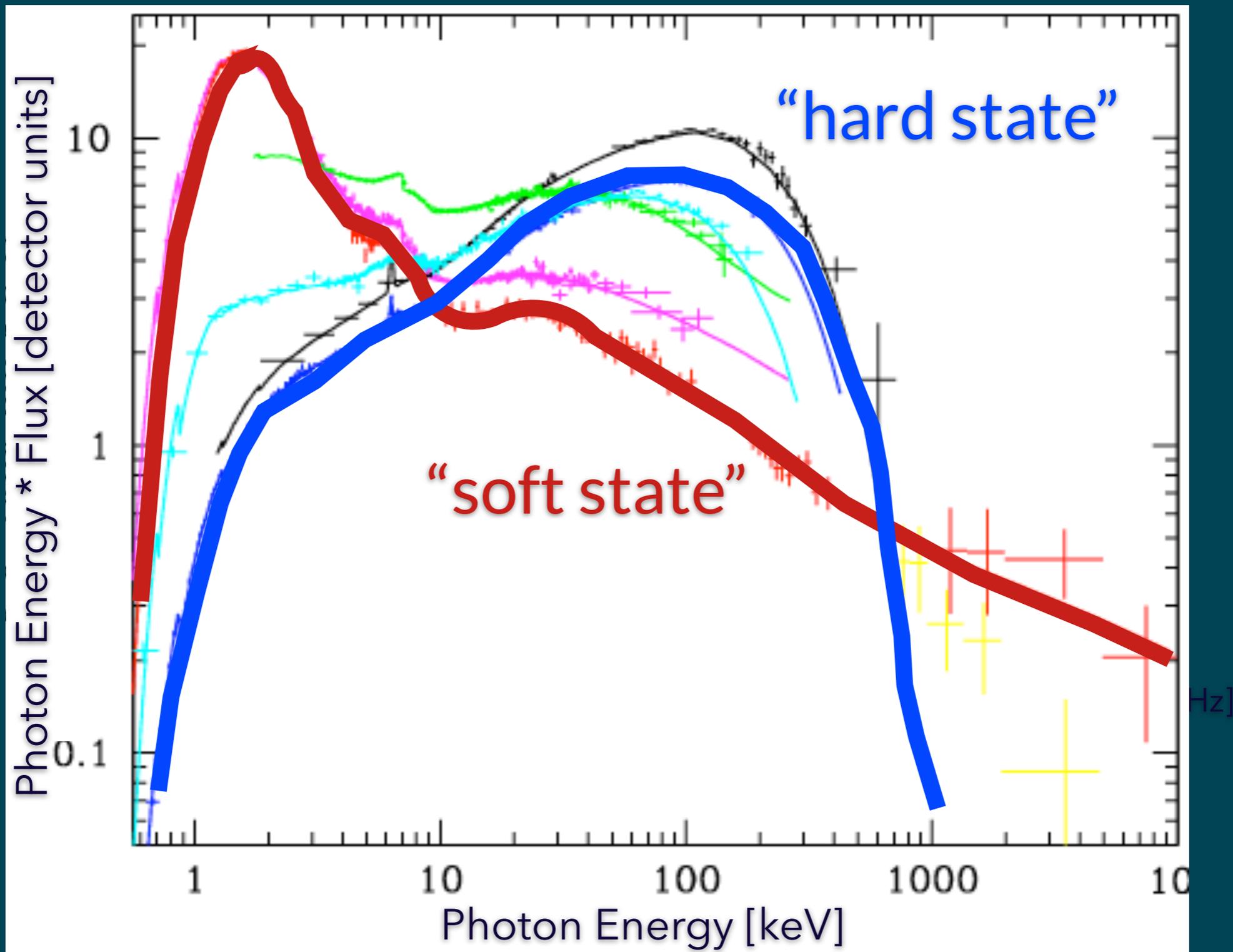
X-ray Spectra Vary with Time



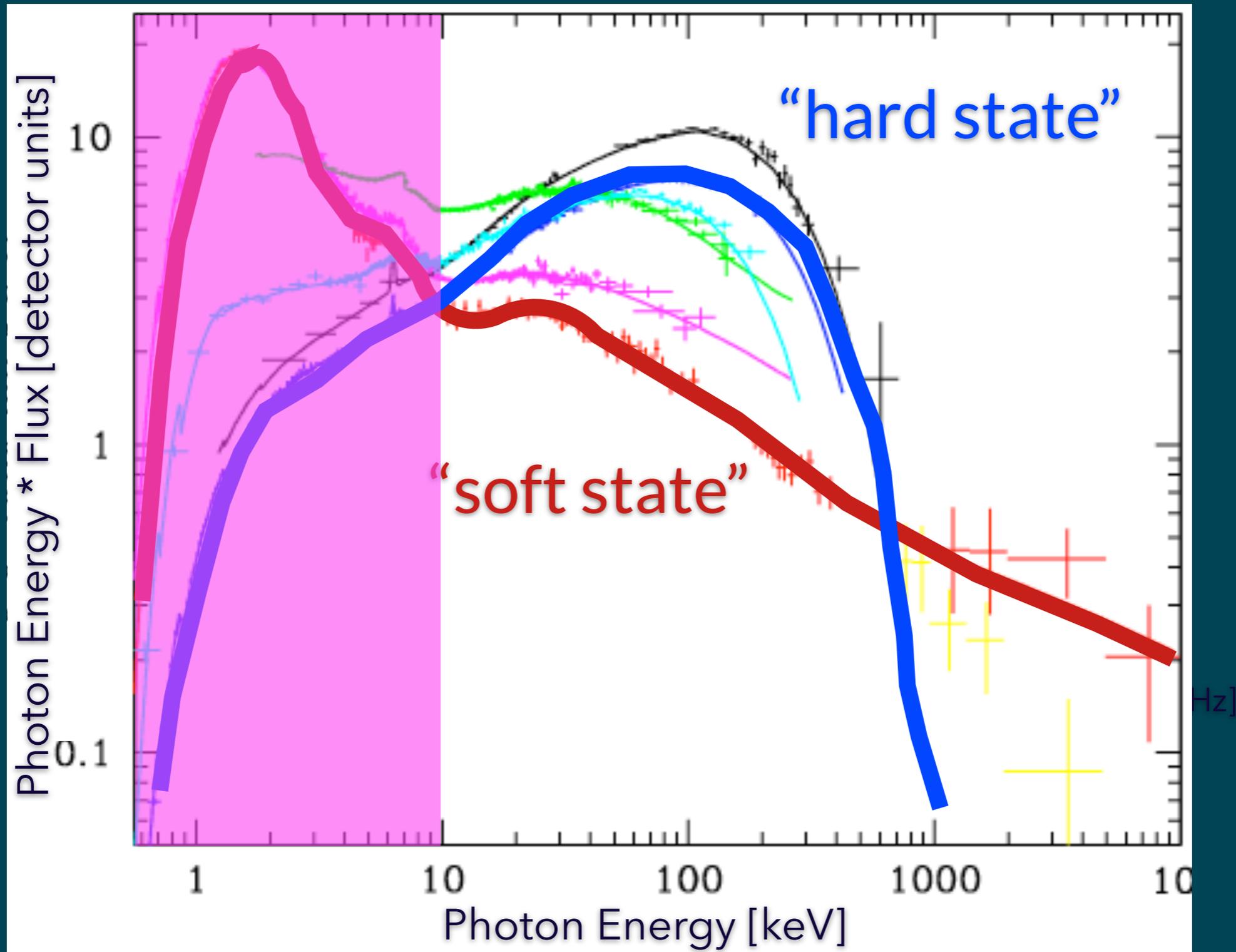
X-ray Spectra Vary with Time



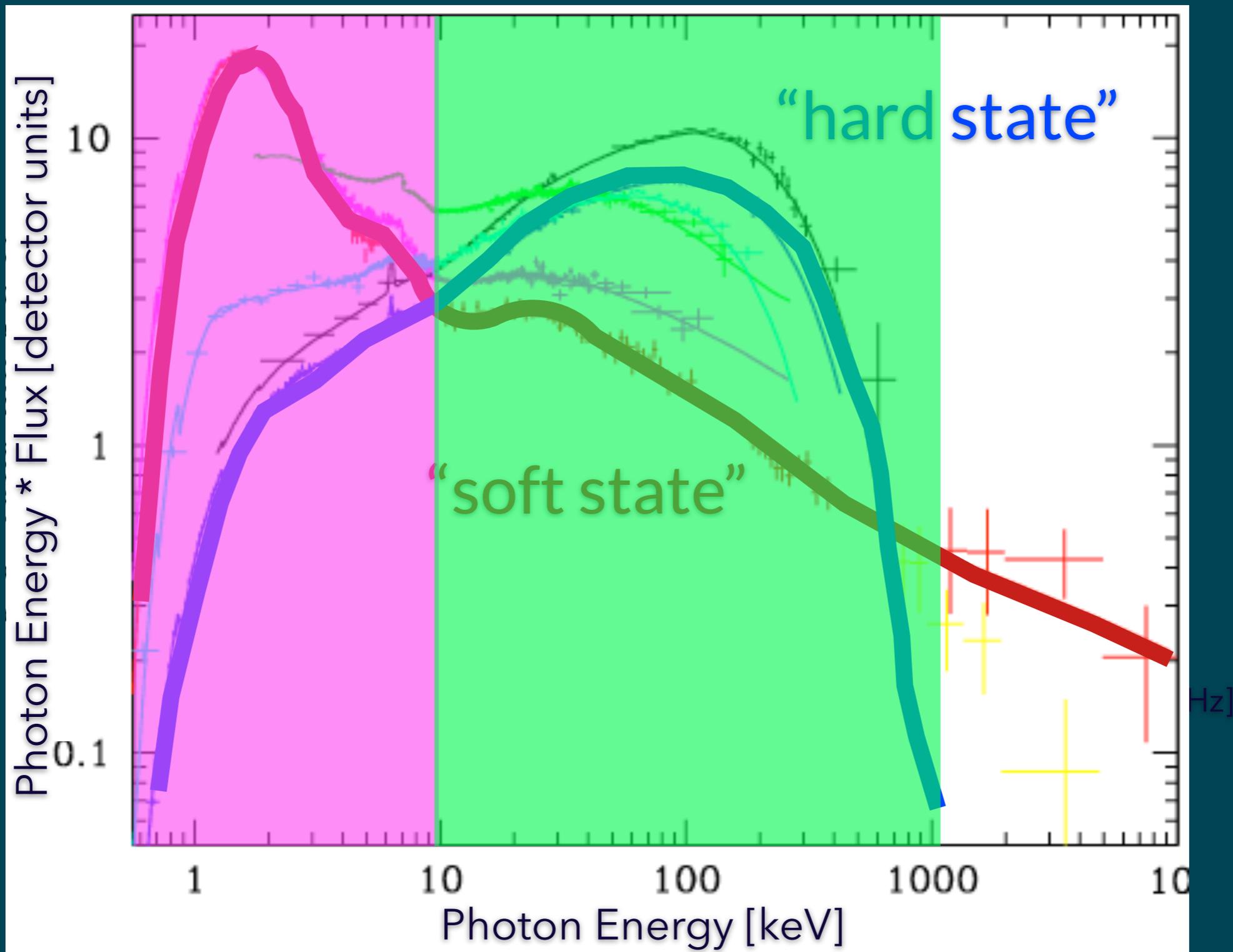
X-ray Spectra Vary with Time



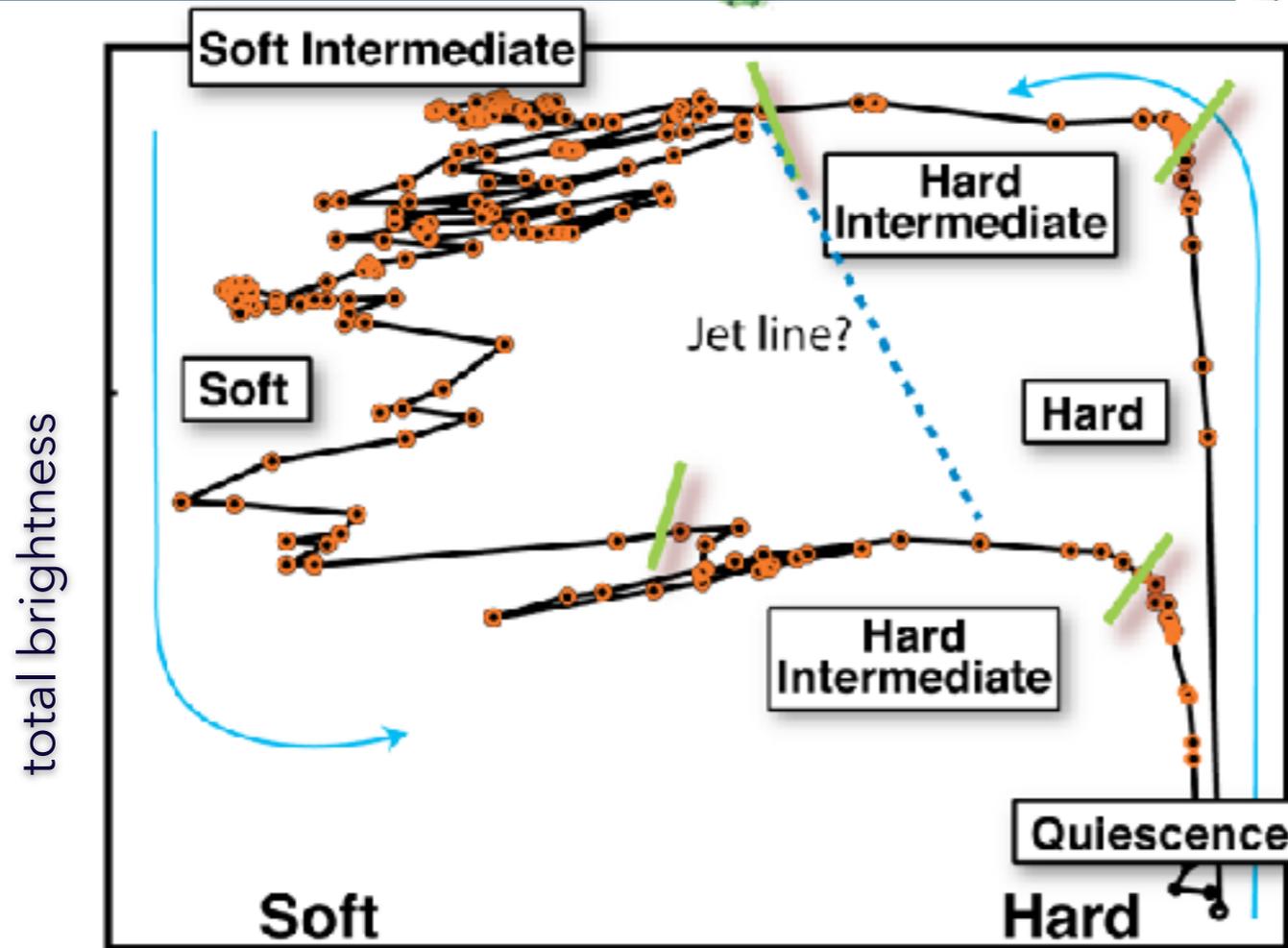
X-ray Spectra Vary with Time



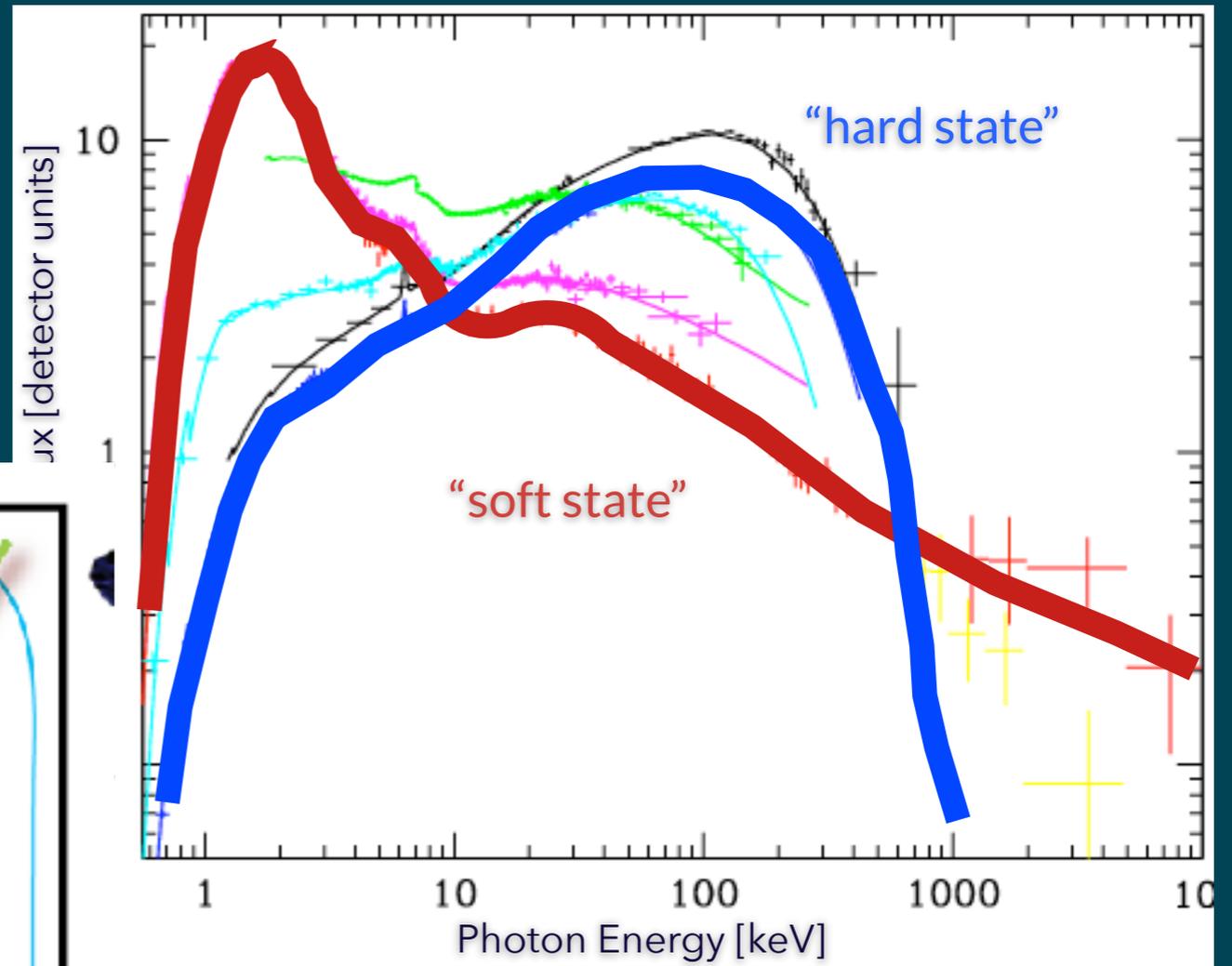
X-ray Spectra Vary with Time



X-ray Spectra Vary with Time

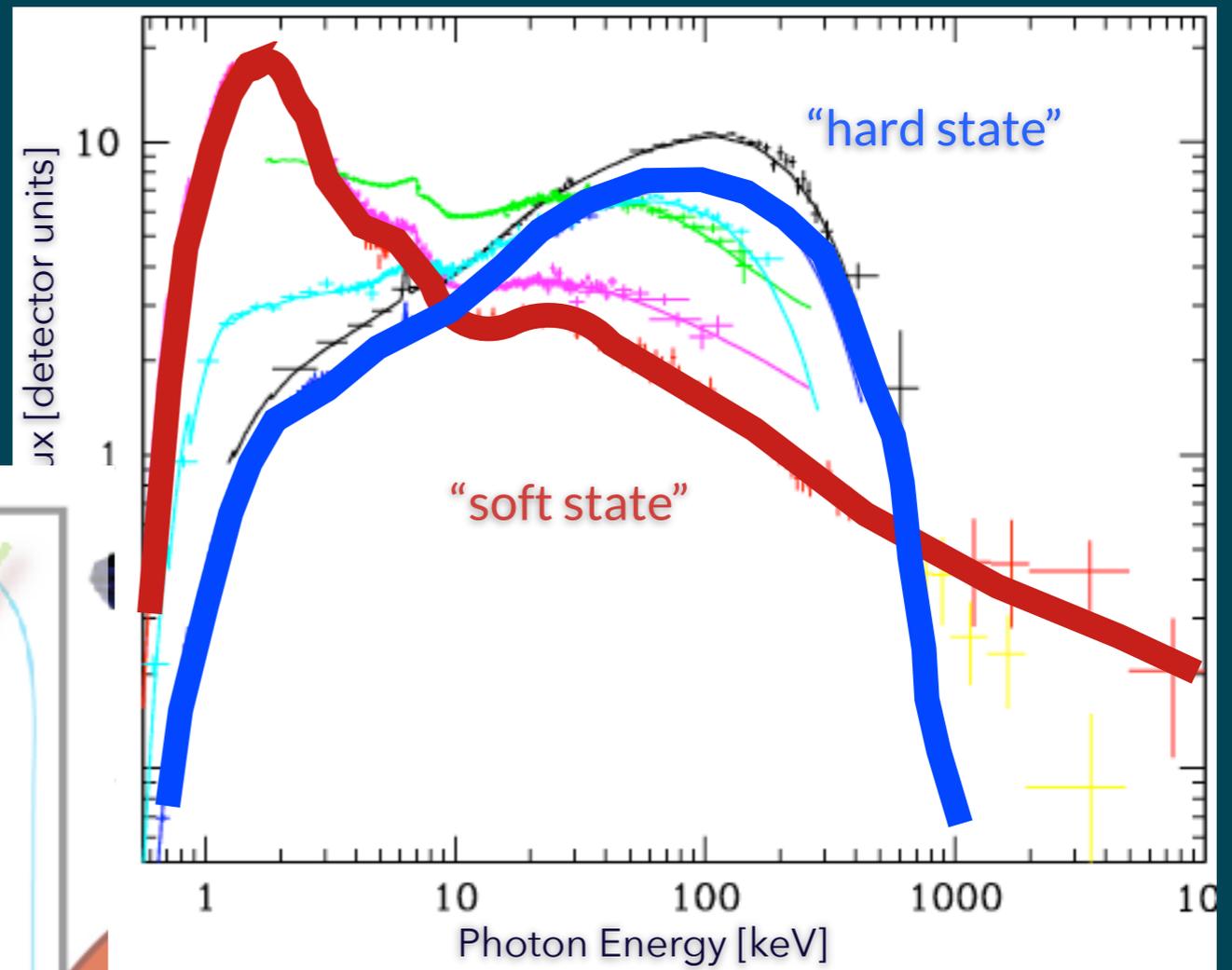
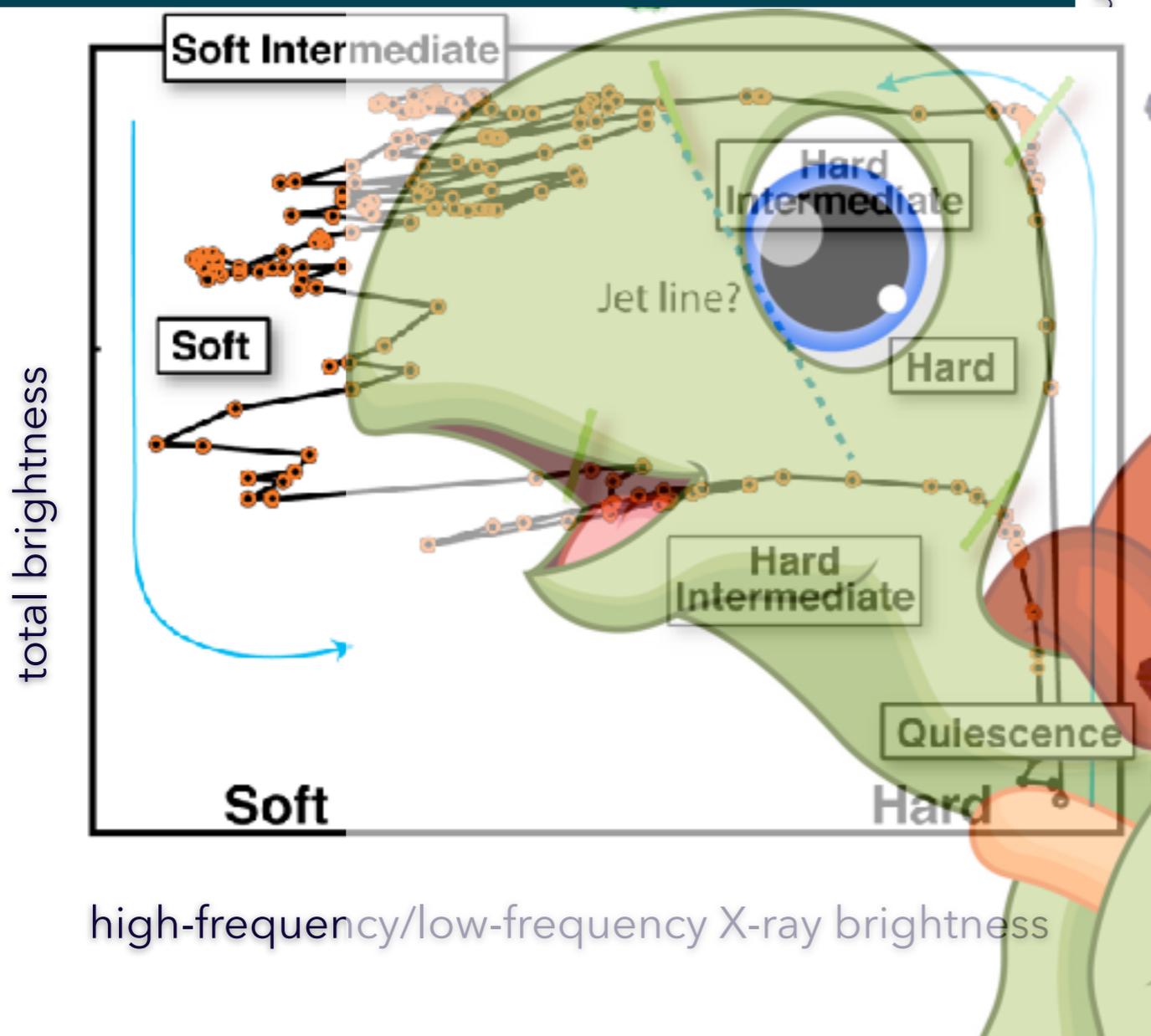


high-frequency/low-frequency X-ray brightness



Malzac (2008)

X-ray Spectra Vary with Time



Malzac (2008)

Fourier Analysis tells us about variability!*

Fourier Analysis tells us about variability!*

***for evenly sampled time series**

Fourier Analysis tells us about variability!*

*for evenly sampled time series

$$x(t) = \frac{1}{N} \sum_j a_j \cos(\omega_j t - \phi_j) = \frac{1}{N} \sum_j (A_j \cos \omega_j t + B_j \sin \omega_j t).$$

Fourier Analysis tells us about variability!*

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$$a_j = \sum_{k=0}^{N-1} x_k e^{2\pi i j k / N}$$

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$$a = \sqrt{A^2 + B^2}$$

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periodogram:

$$P_j \equiv \frac{2}{N_{ph}} |a_j|^2 \quad j = 0, \dots, \frac{N}{2},$$

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$$a_j = \sum_{k=0}^{N-1} x_k e^{2\pi i j k / N}$$

useful normalization



periodogram:

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periodogram:

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statistical distribution?

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 assume many data points

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assume many data points

~ Gaussian

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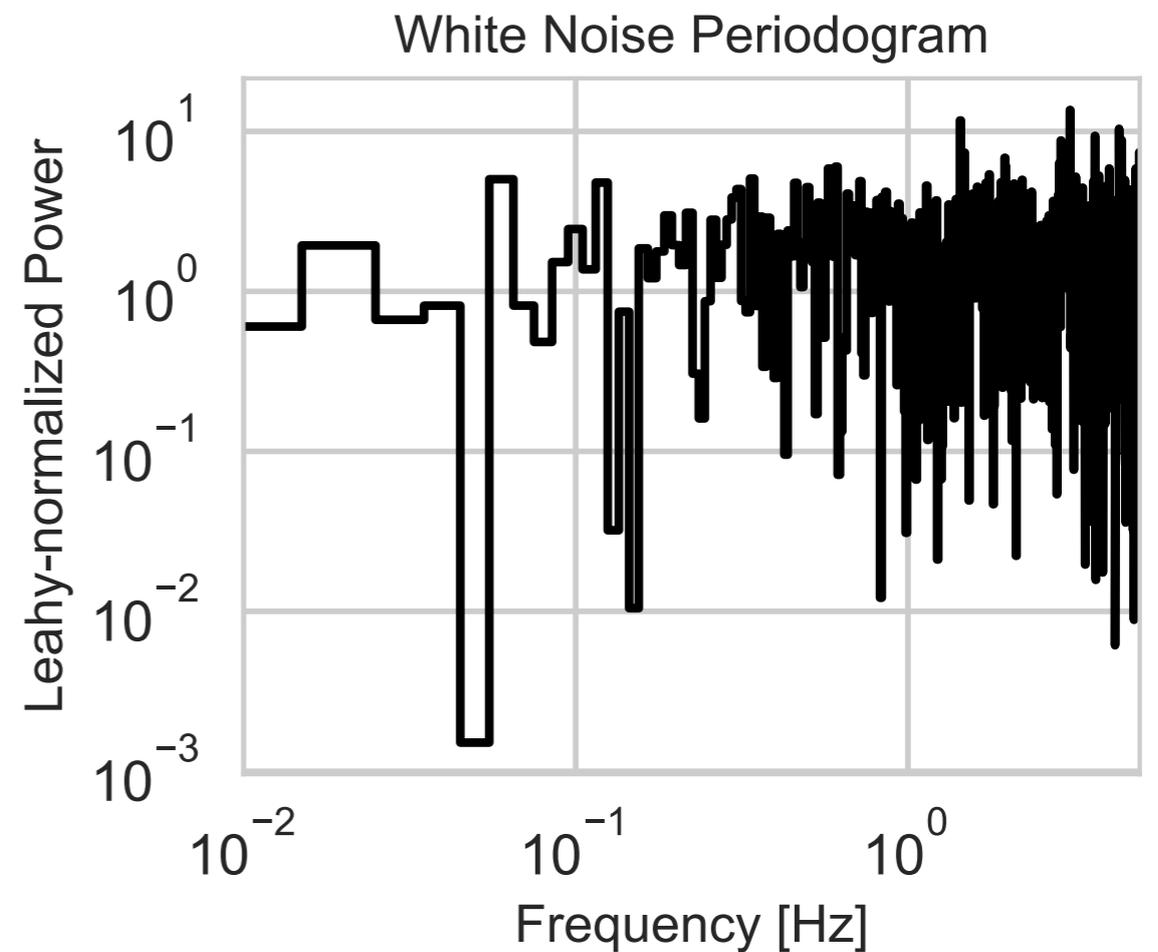
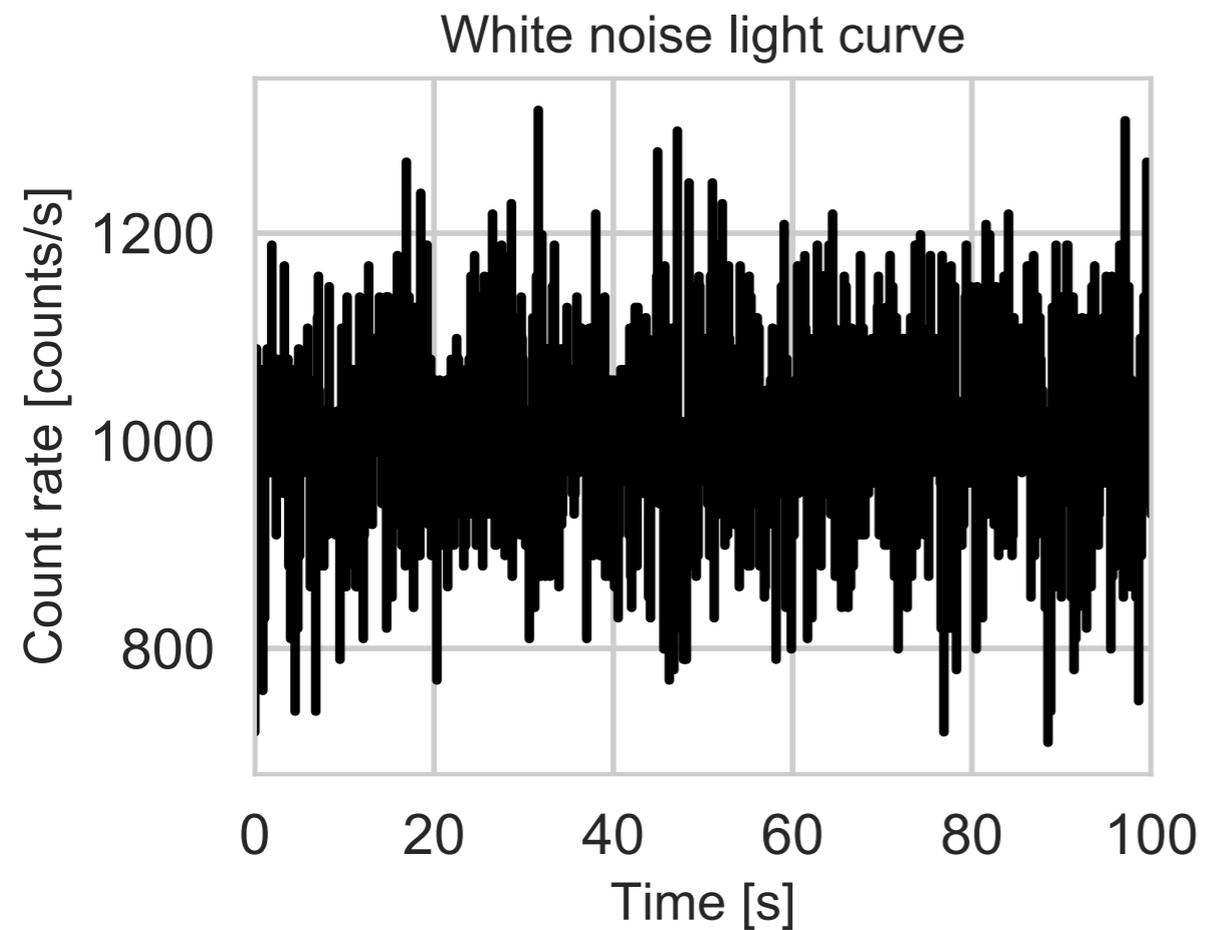
$$P_j \equiv \frac{2}{N_{ph}} |a_j|^2 \quad j = 0, \dots, \frac{N}{2},$$

$$|a_j|^2 = A^2 + B^2$$

χ^2 with 2 degrees of freedom

Fourier Analysis tells us about variability!*

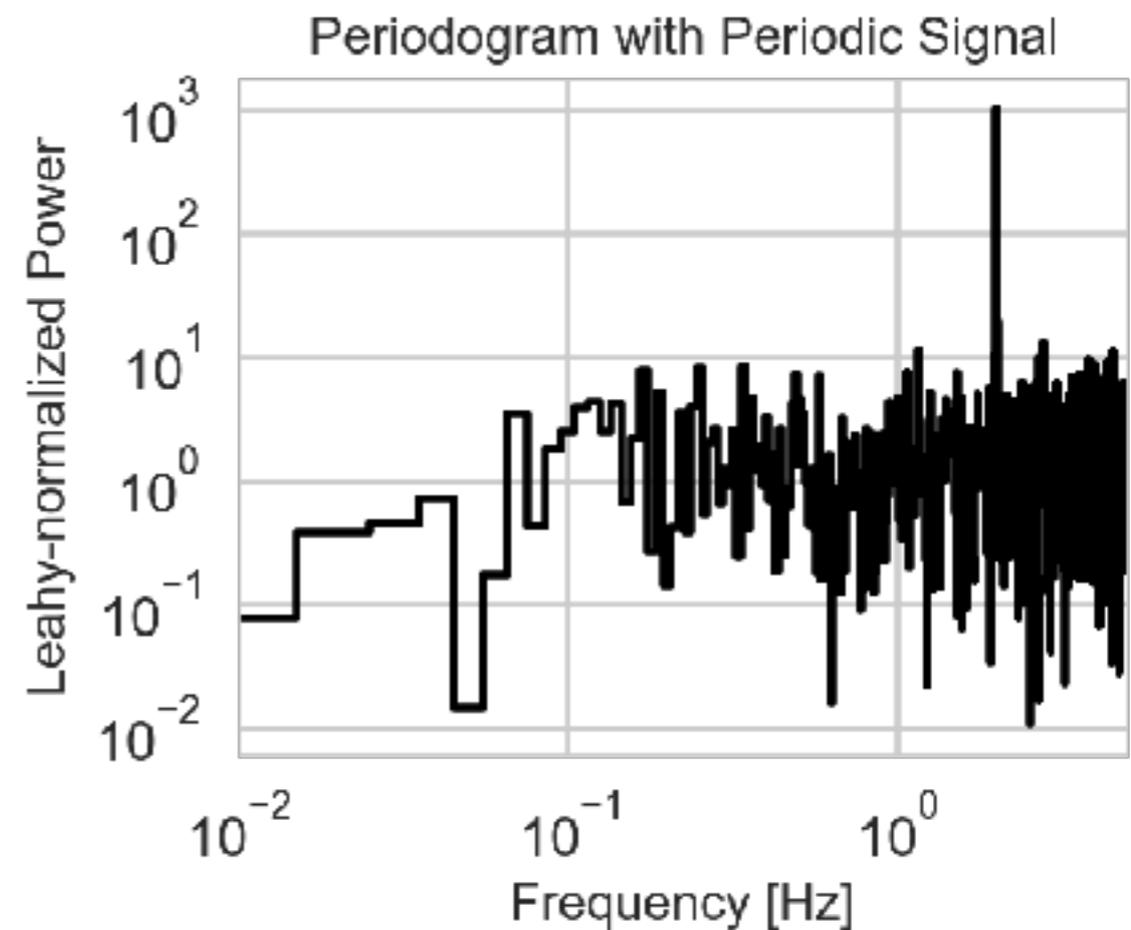
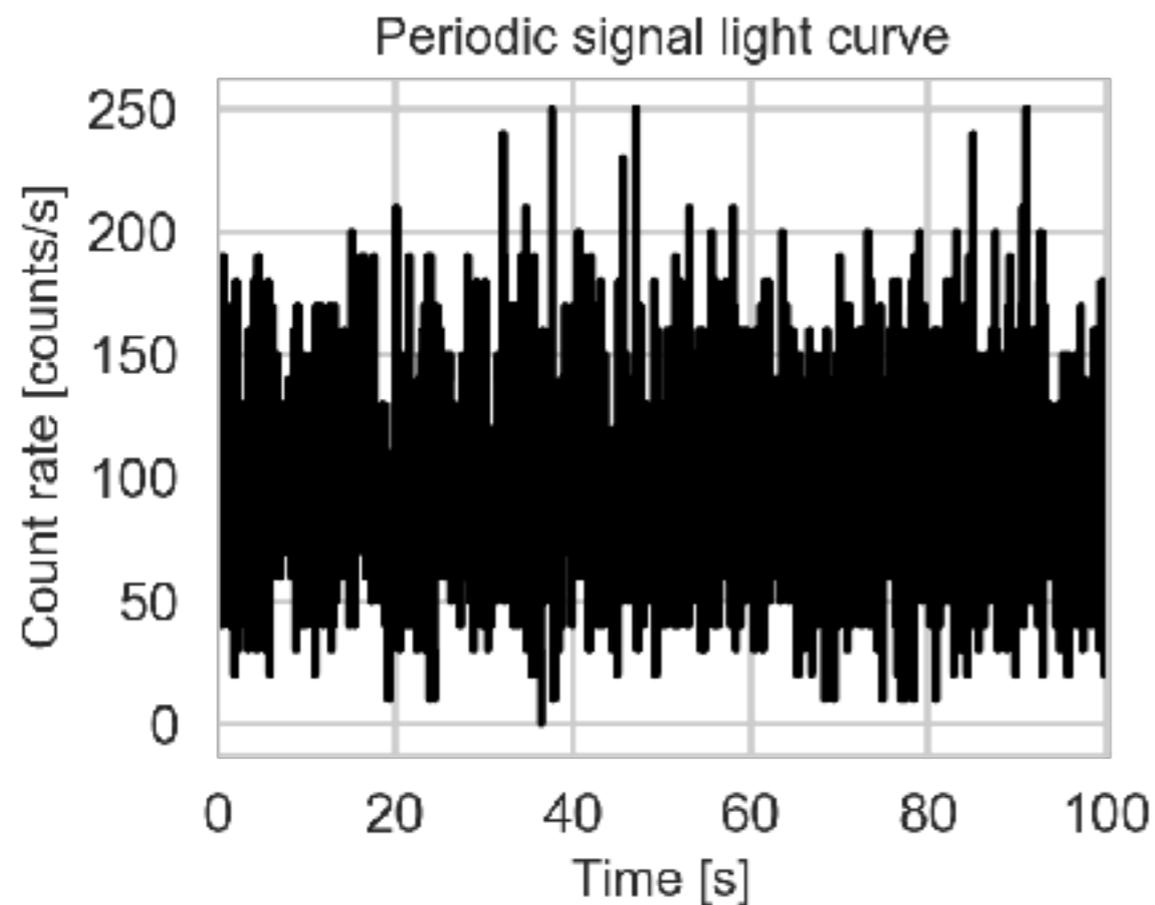
*for evenly sampled time series



white noise

Fourier Analysis tells us about variability!*

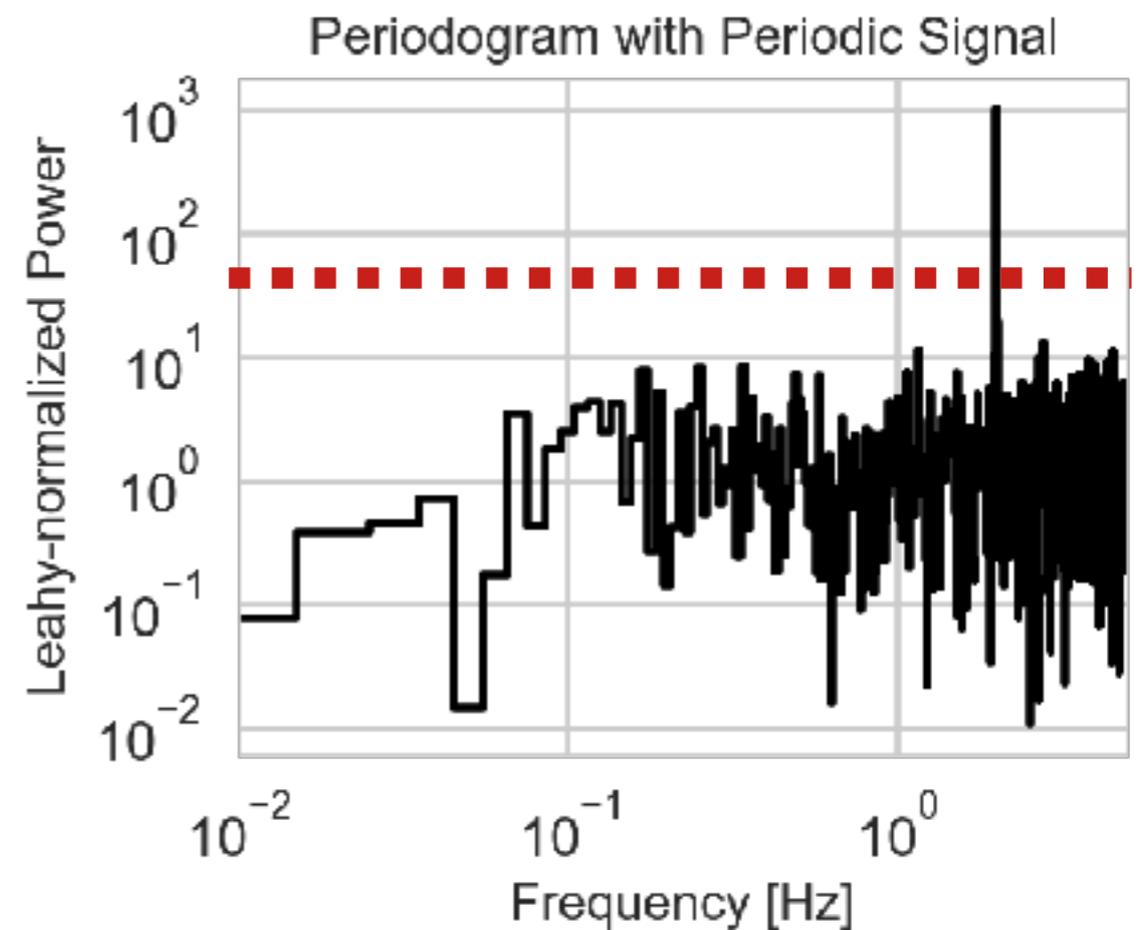
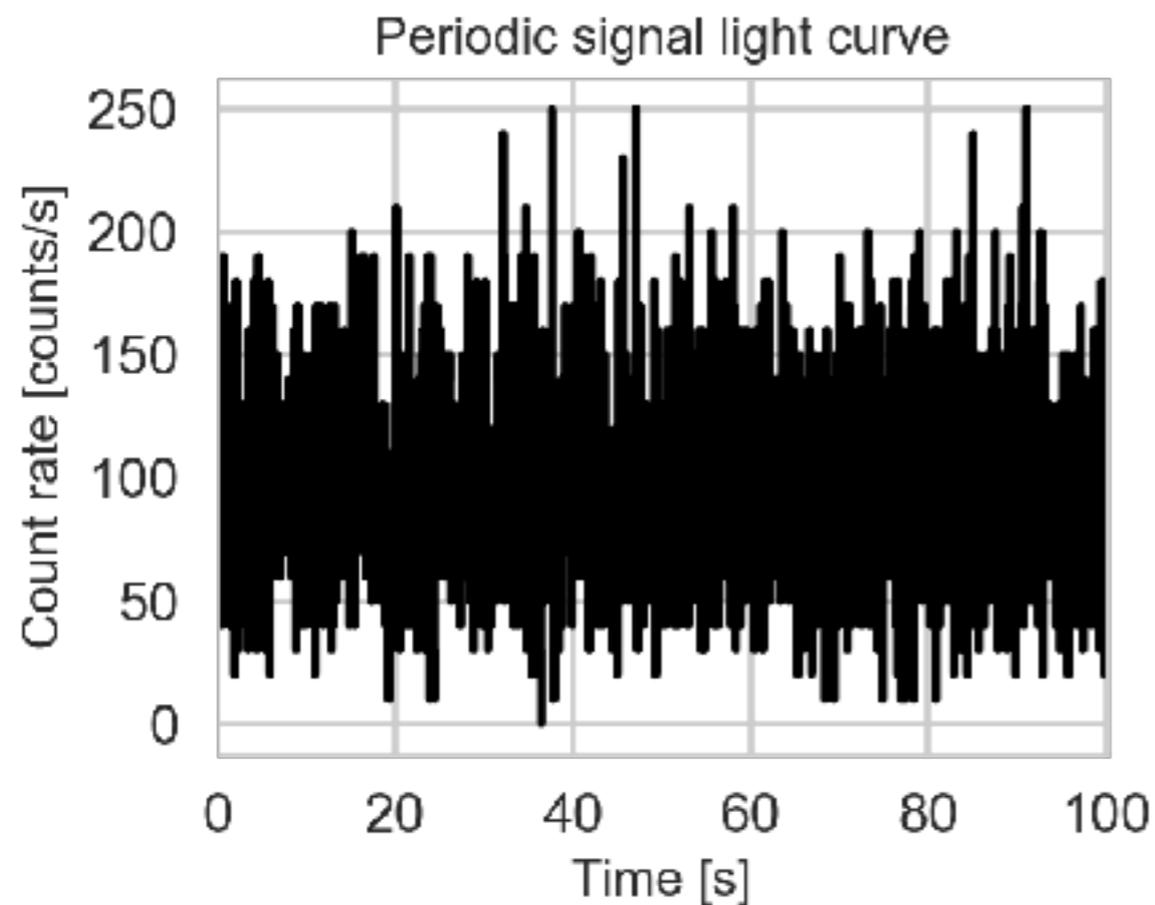
*for evenly sampled time series



periodic signal

Fourier Analysis tells us about variability!*

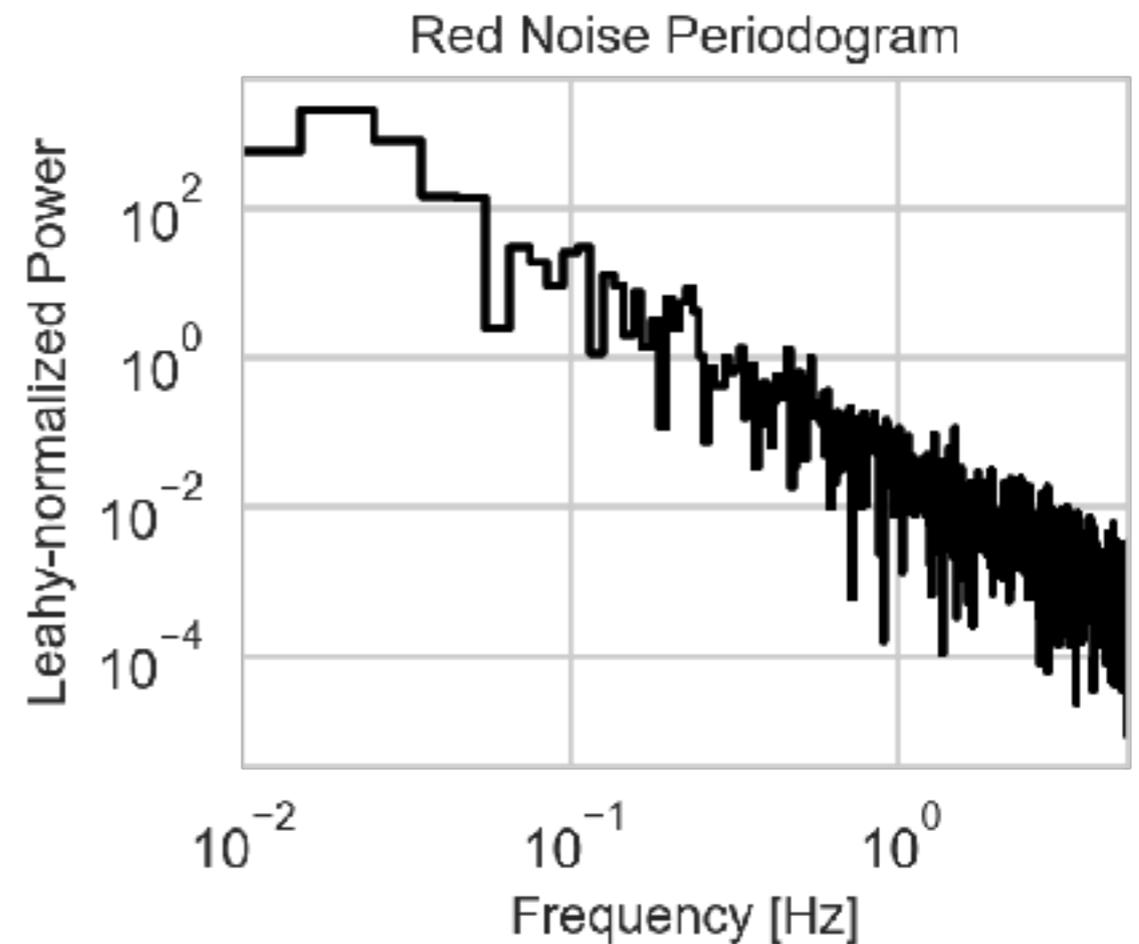
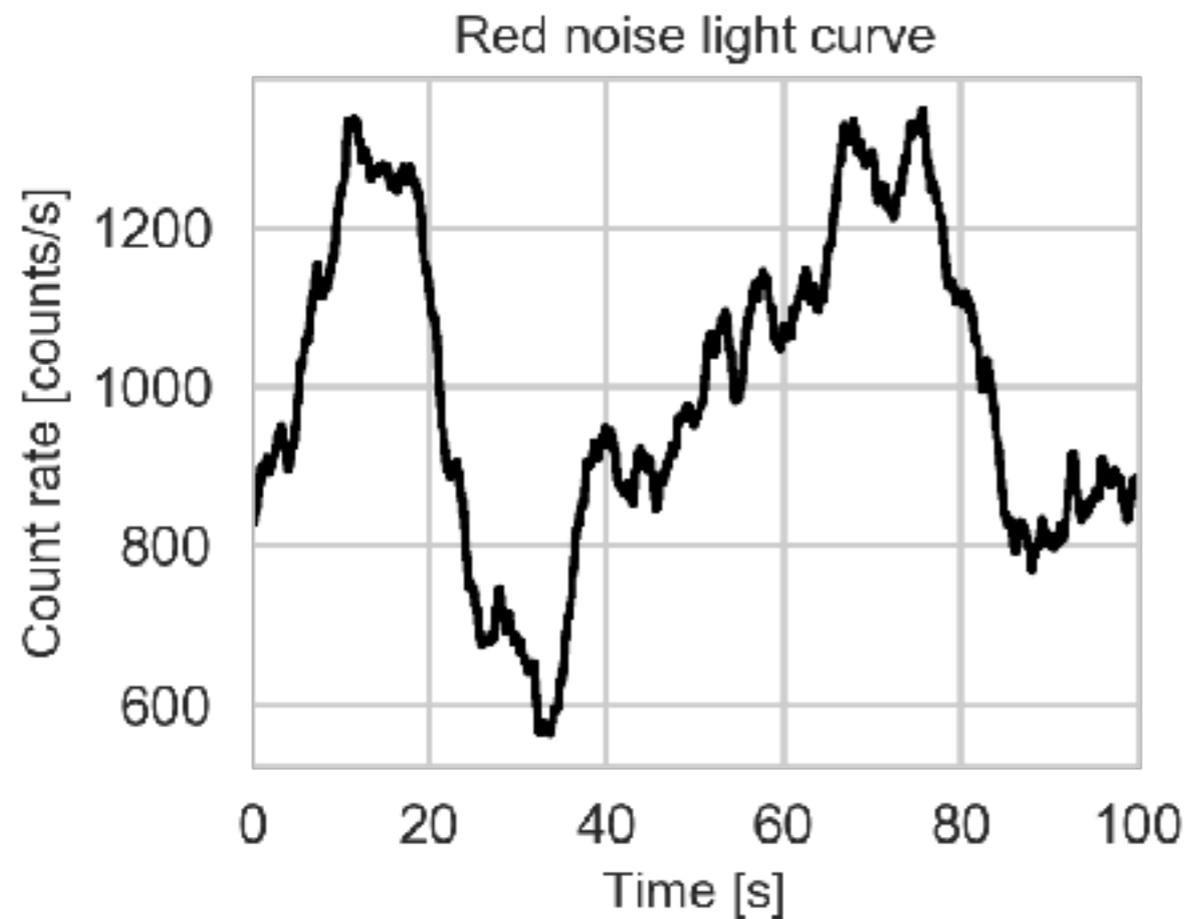
*for evenly sampled time series



periodic signal

Fourier Analysis tells us about variability!*

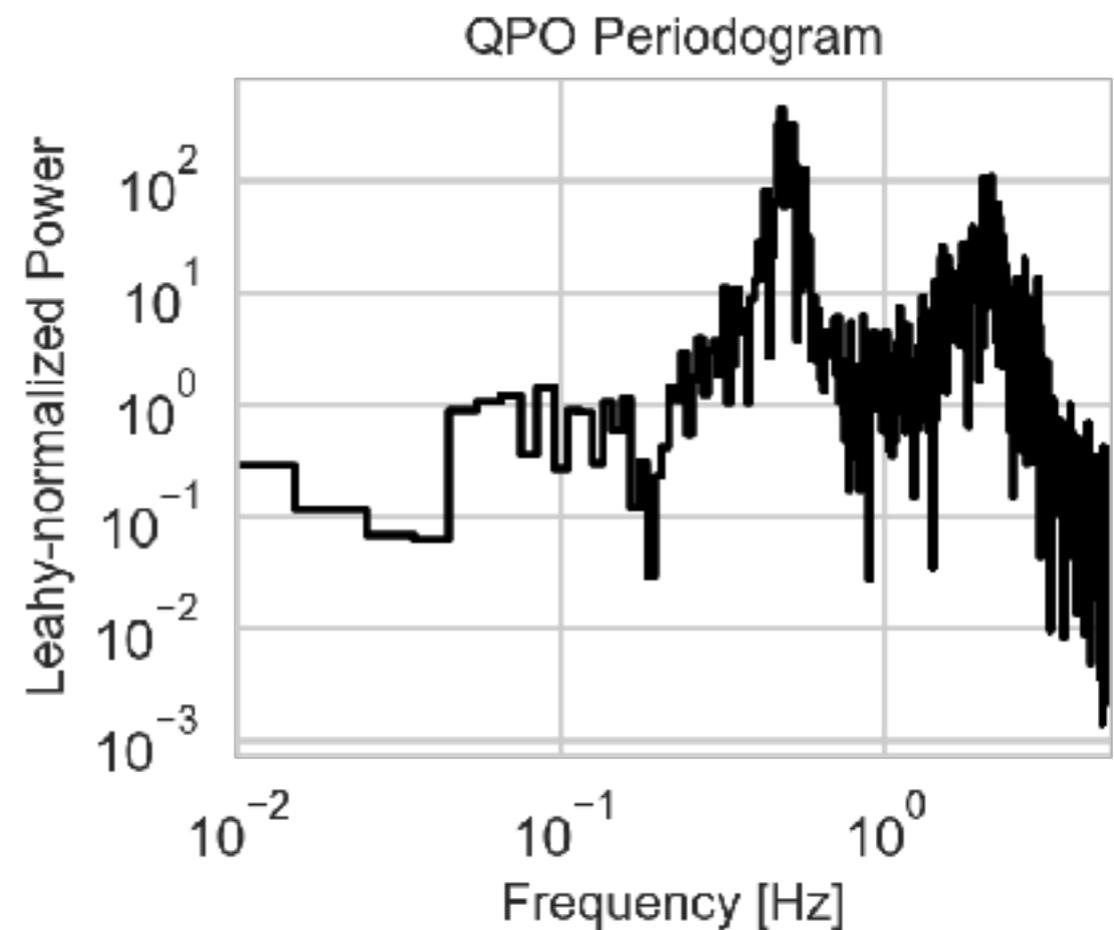
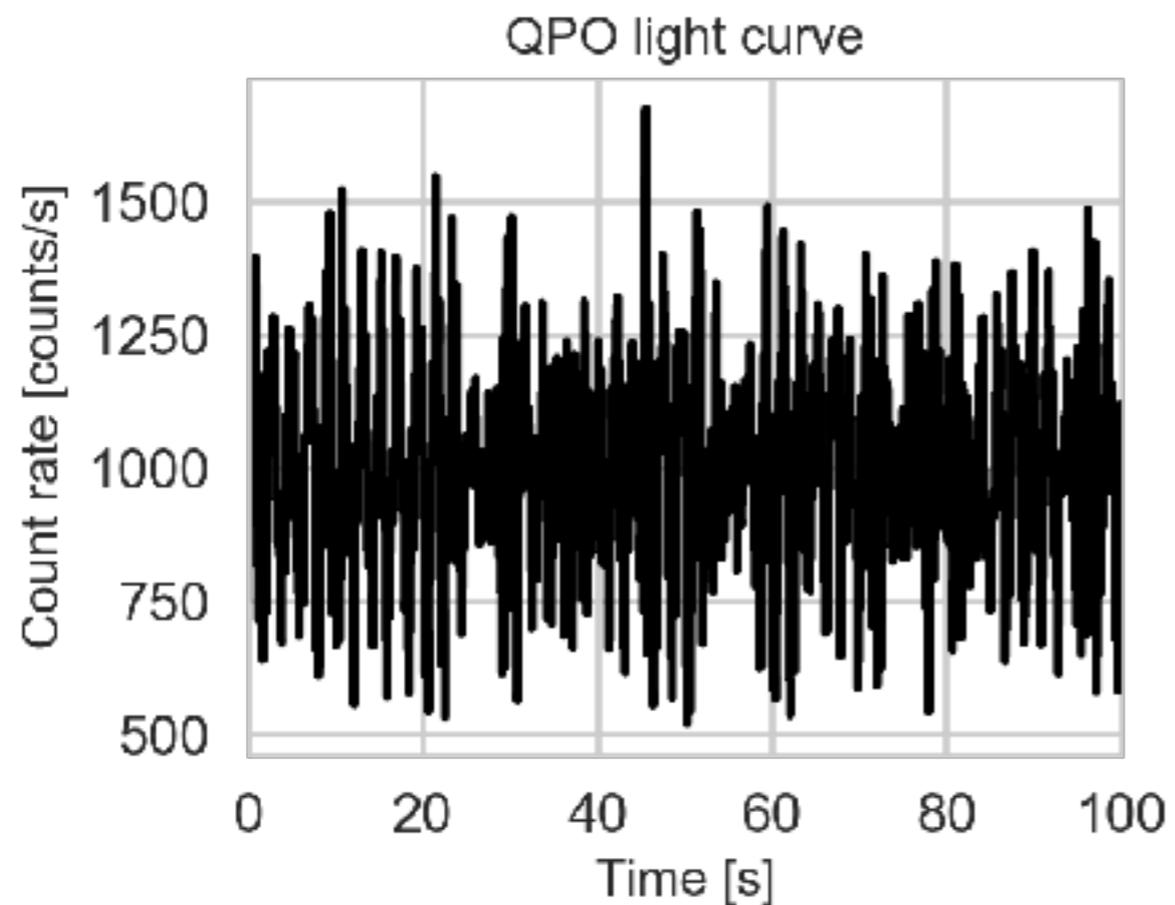
*for evenly sampled time series



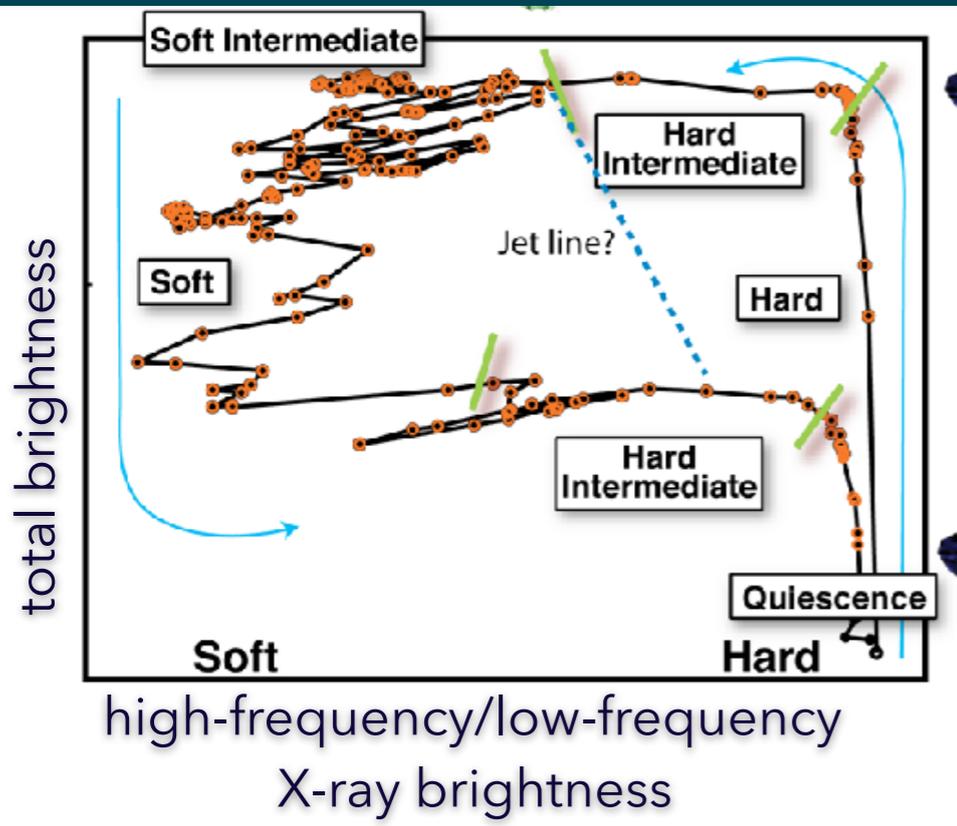
red noise

Fourier Analysis tells us about variability!*

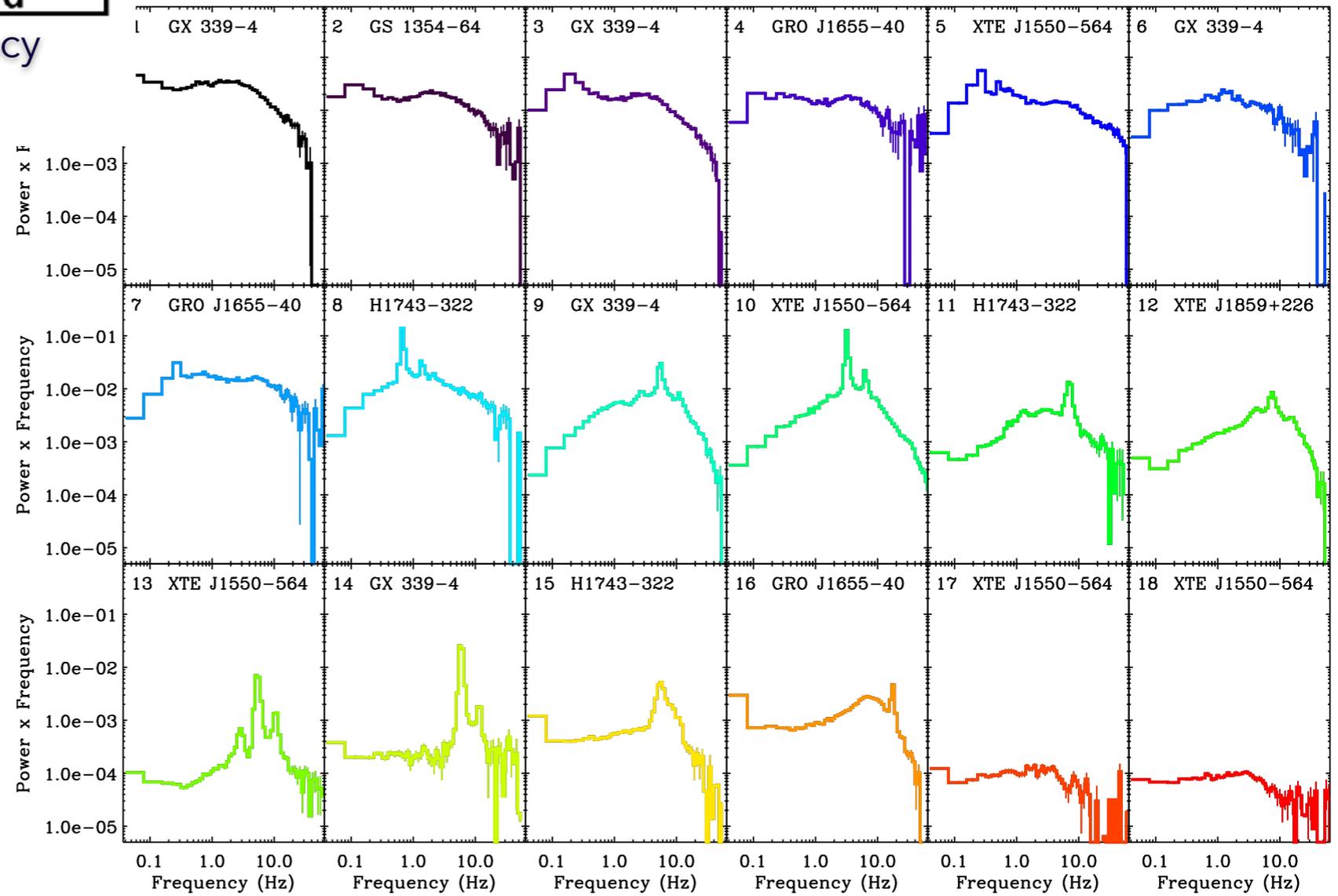
*for evenly sampled time series



quasi-periodic oscillations



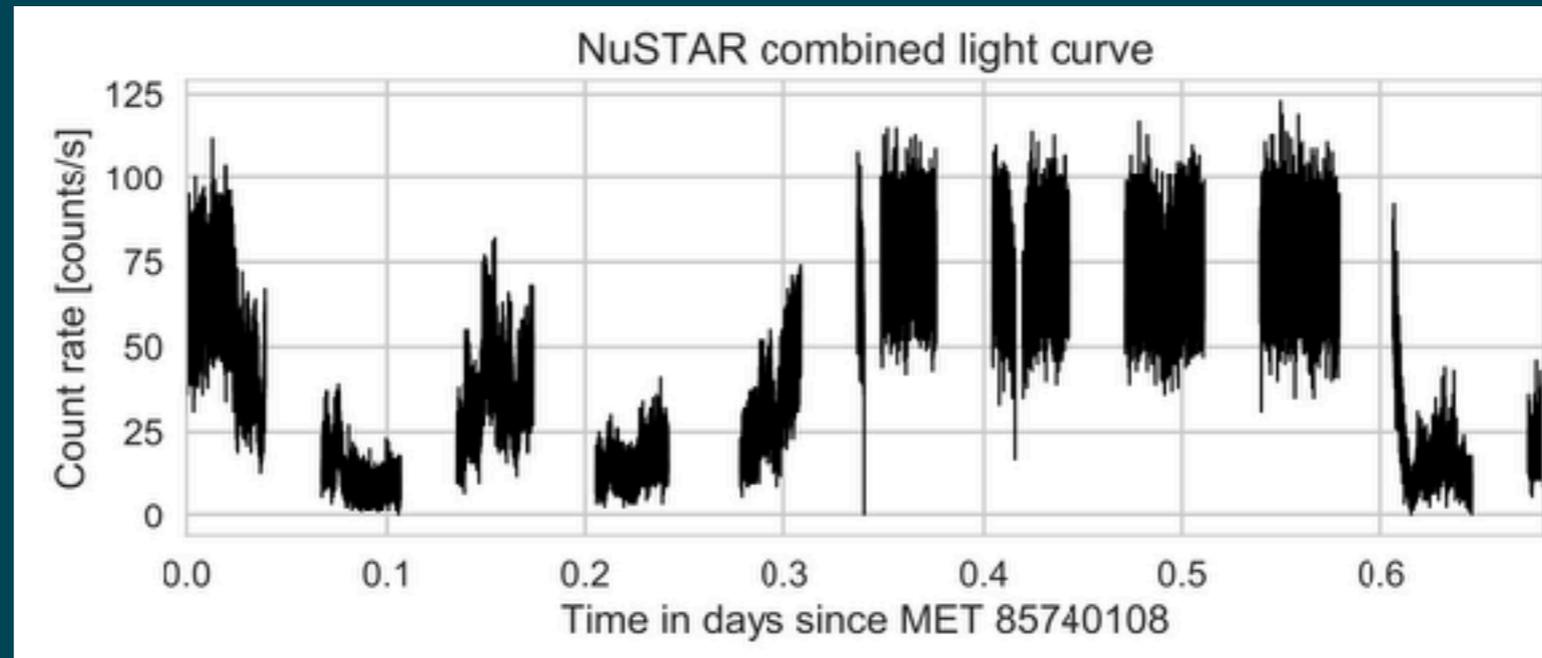
Credit: Sera Markoff



So, we're done, right?

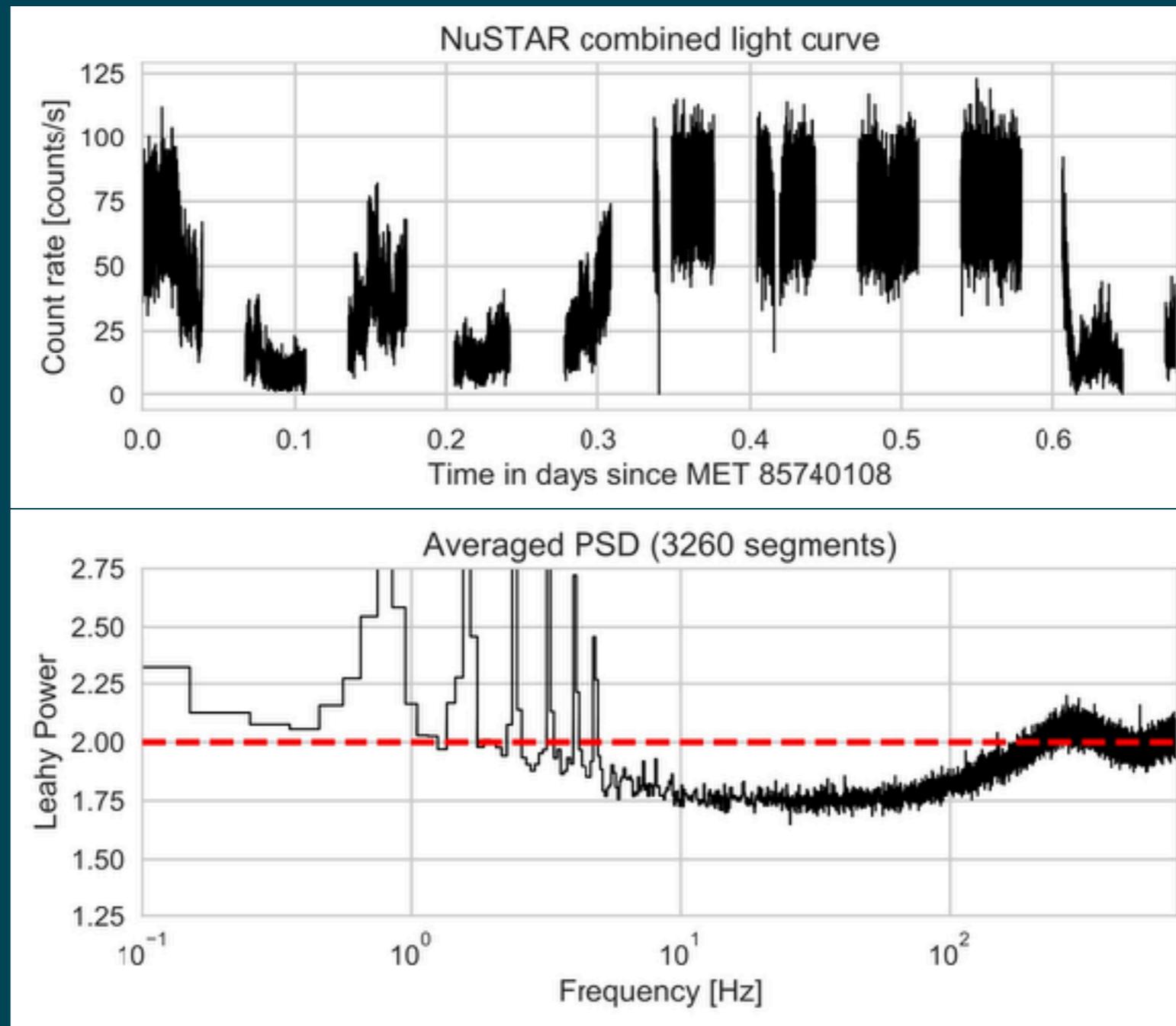
Hercules X-1

Huppenkothen & Bachetti, in press



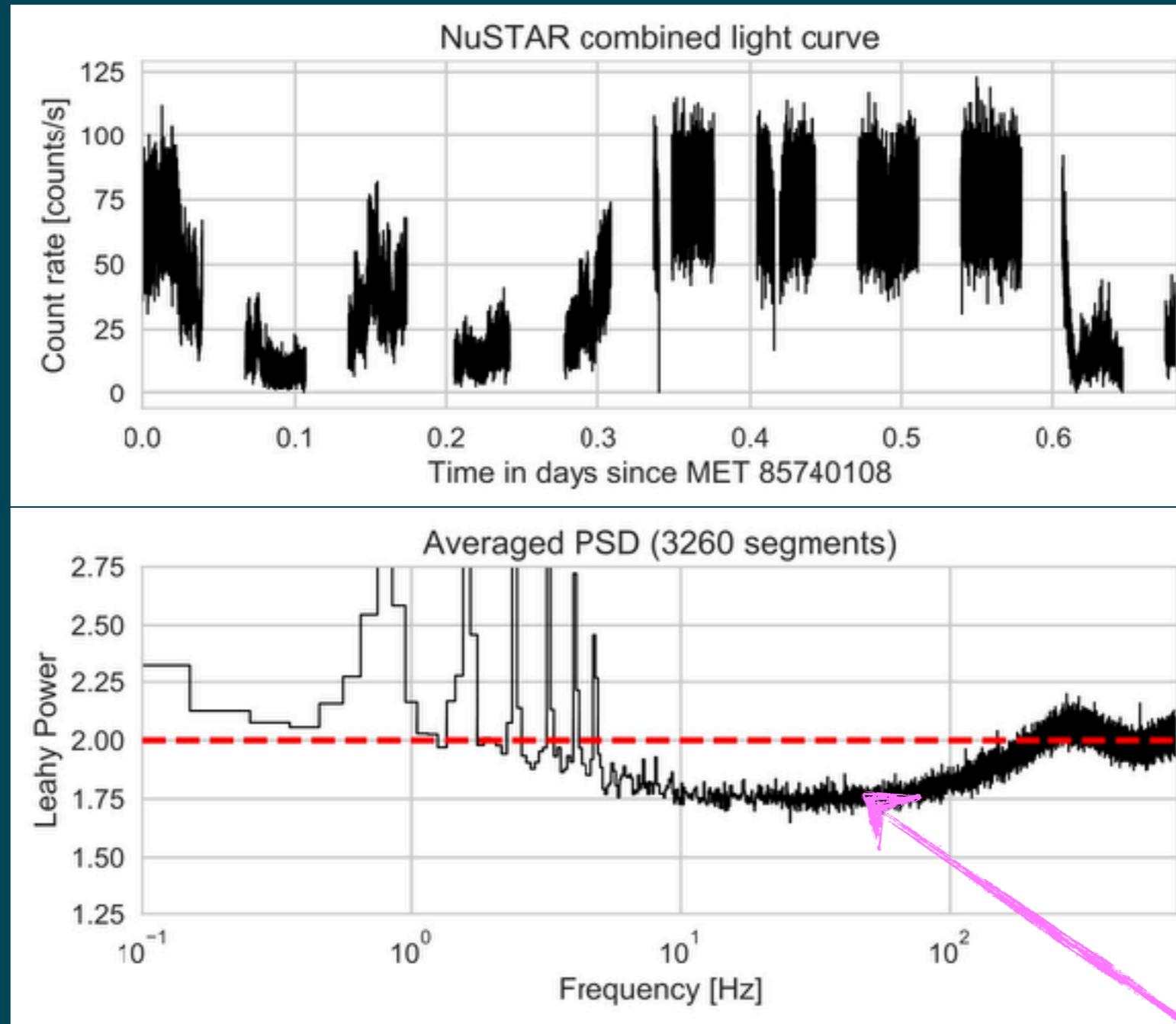
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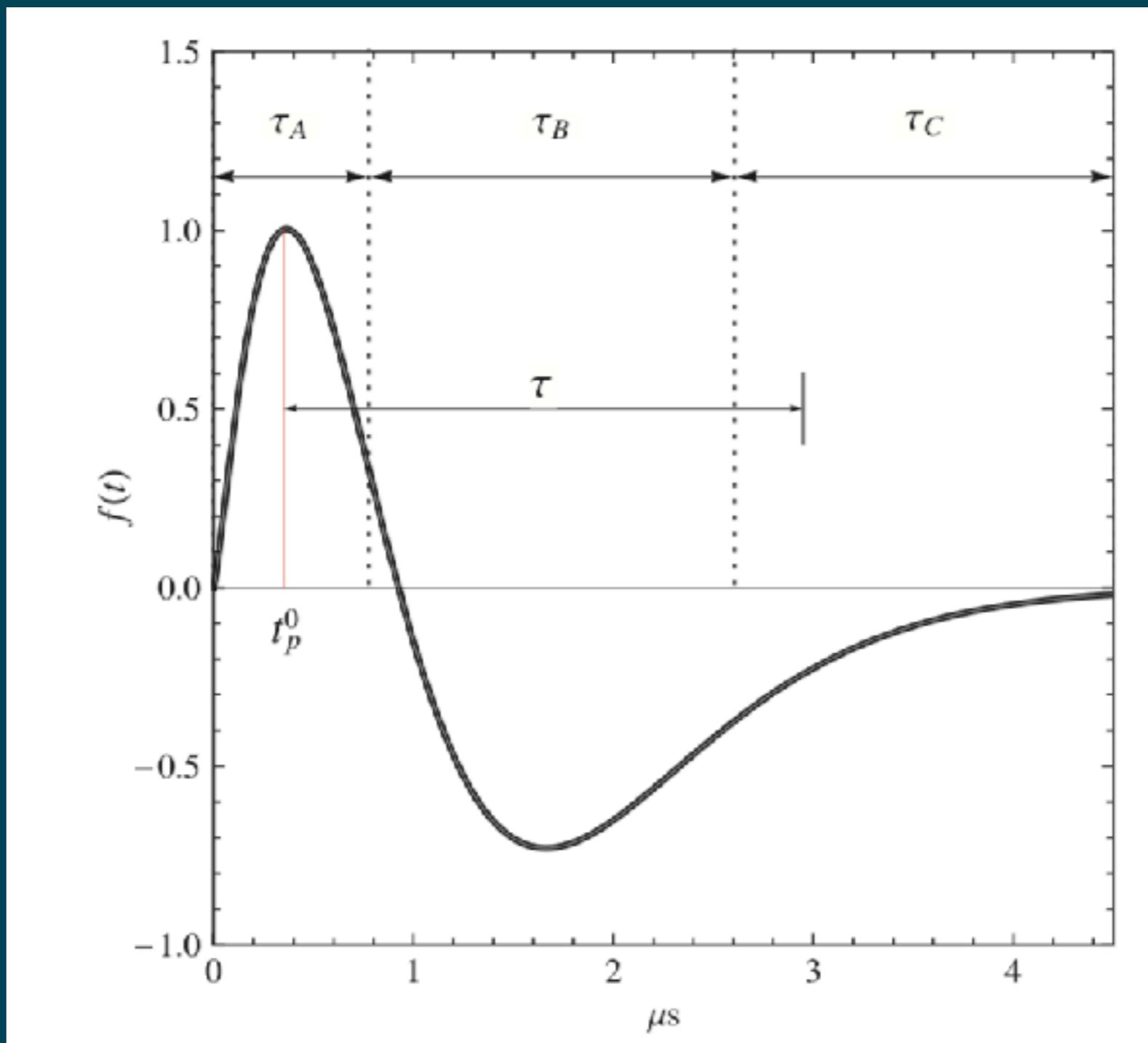
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Huppenkothen & Bachetti, in press

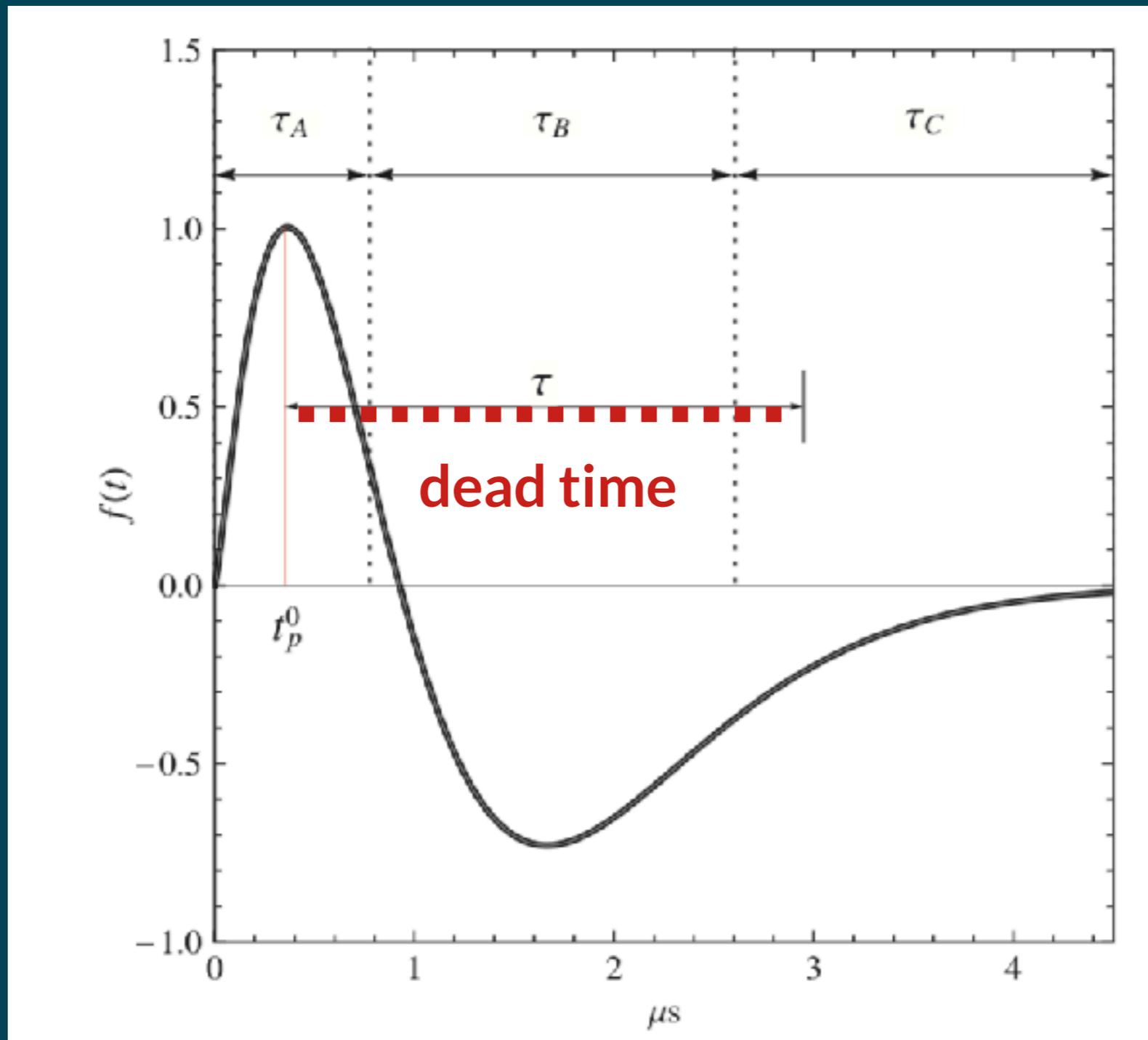


???

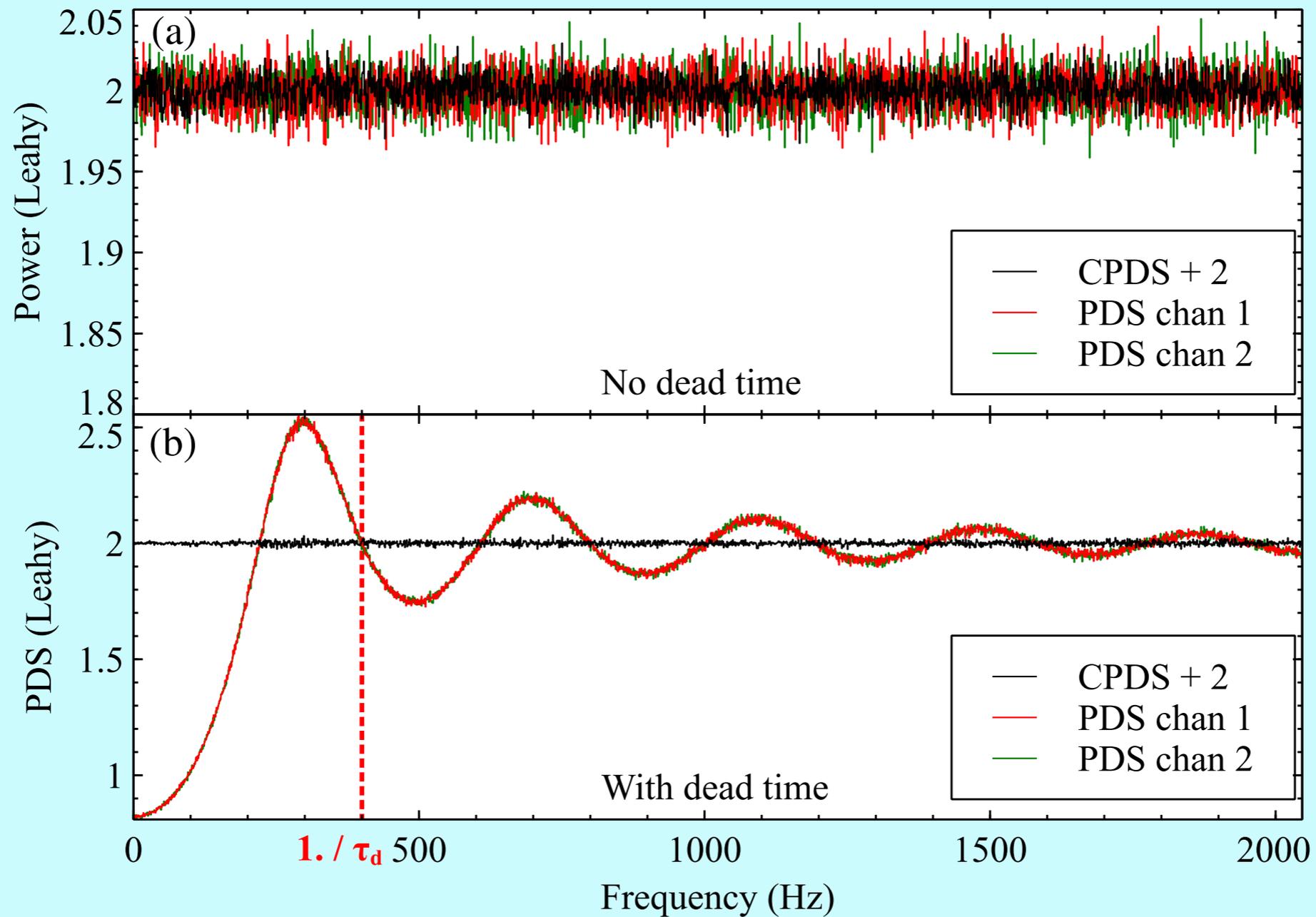
After **detection of a photon**, the detector is
“**dead**” for a short interval



After **detection of a photon**, the detector is
“**dead**” for a short interval



Dead time affects mean and variance of the periodogram

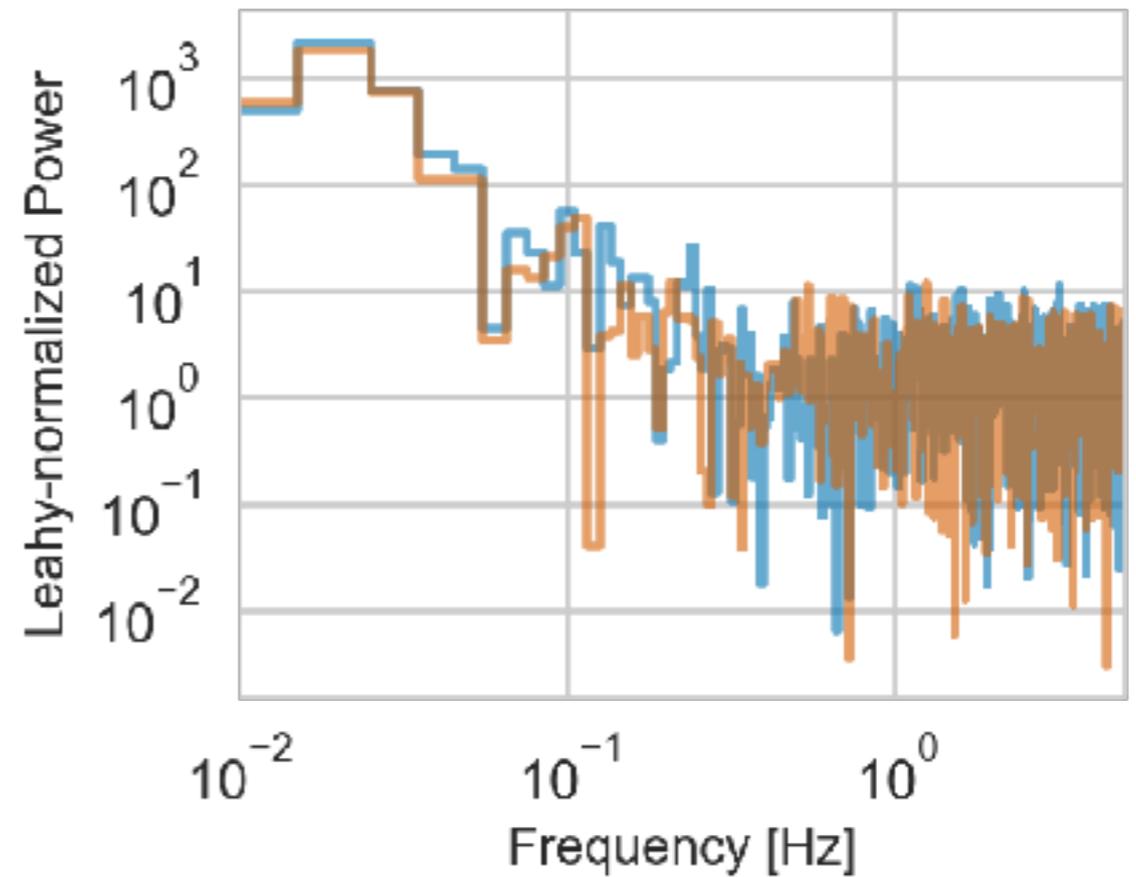
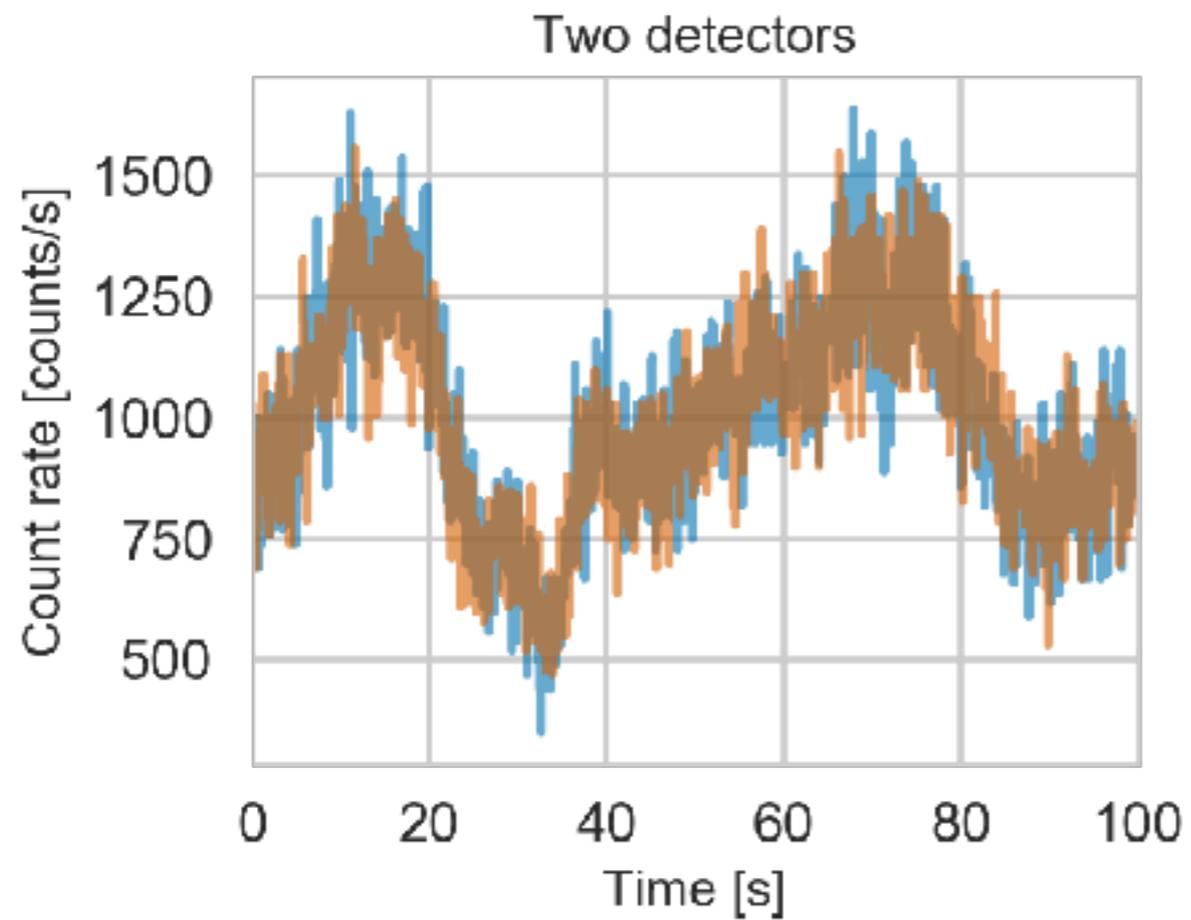


Can we fix this?

Yes (ish)



Having Two Detectors Helps!



Signal is the same, but the measurement noise is different!

Compute cross spectrum instead of periodogram

$$\mathcal{F}_x(j)\mathcal{F}_y^*(j) = \frac{1}{2} \{ (A_{xj}A_{yj} + B_{xj}B_{yj}) + i(A_{xj}B_{yj} - A_{yj}B_{xj}) \}$$

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phase/time lag

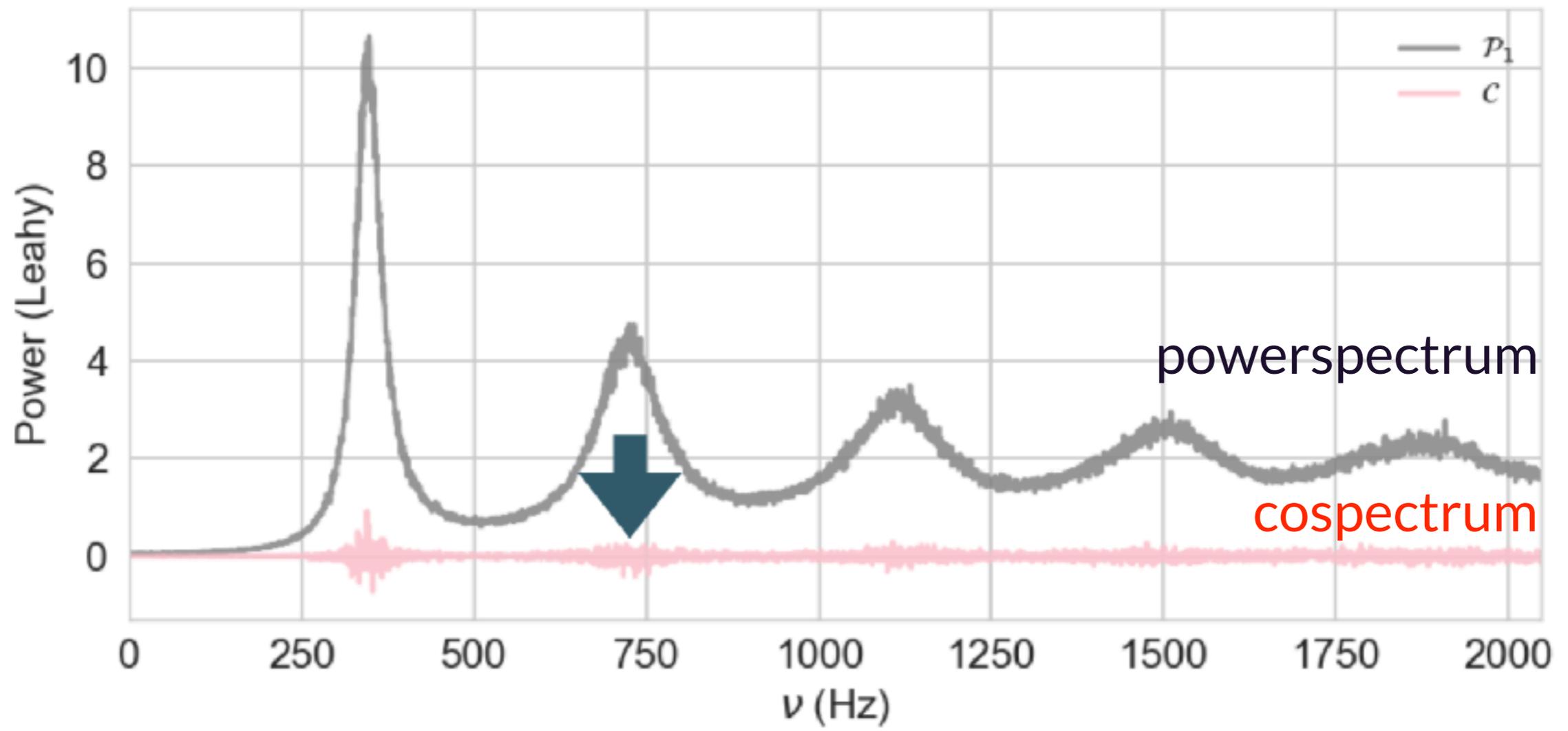
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phase/time lag

$$C_j = \frac{1}{2} (A_{xj}A_{yj} + B_{xj}B_{yj}) .$$

cospectrum



**What is the statistical
distribution of the cospectrum?**

periodogram

periodogram

$$|a_j|^2 = A^2 + B^2$$

periodogram

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χ^2 distributed

periodogram

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χ^2 distributed

cospectrum

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periodogram

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χ^2 distributed

cospectrum

$$C_j = \frac{1}{2}(A_{xj}A_{yj} + B_{xj}B_{yj}).$$

$$A_{xj}^2 \neq A_{xj}A_{yj}$$

periodogram

$$|a_j|^2 = A^2 + B^2$$



χ^2 distributed

cospectrum

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$$A_{xj}^2 \neq A_{xj}A_{yj}$$



not χ^2 distributed

we know

$$A_{xj}, B_{xj} \sim \mathcal{N}(0, \sigma_x^2)$$

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set

$$Z = A_{xj}A_{yj} , Q = B_{xj}B_{yj}$$

we know

$$A_{xj}, B_{xj} \sim \mathcal{N}(0, \sigma_x^2)$$

set

$$Z = A_{xj}A_{yj}, \quad Q = B_{xj}B_{yj}$$

then the PDF is:

$$P_Z(z) = \frac{K_0\left(\frac{|z|}{\sigma_x \sigma_y}\right)}{\pi \sigma_x \sigma_y},$$

Watson (1922); Wishart
& Bartlett (1932)

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Bessel
function



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the distribution for

$$Z + Q$$

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set

$$Z = A_{xj}A_{yj}, \quad Q = B_{xj}B_{yj}$$

Bessel
function



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Watson (1922); Wishart
& Bartlett (1932)

the distribution for

$$Z + Q$$

is the convolution of PDFs:

$$p_{Z+Q}(z) = p_Z * p_Q(z)$$

combine

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$$p_{Z+Q}(z) = p_Z * p_Q(z)$$

and

$$P_Z(z) = \frac{K_0 \left(\frac{|z|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y},$$

combine

$$p_{Z+Q}(z) = p_Z * p_Q(z)$$

and

$$P_Z(z) = \frac{K_0 \left(\frac{|z|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y},$$

to get

$$p_{Z+Q}(z) = \int_{-\infty}^{+\infty} \frac{K_0 \left(\frac{|t|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y} \frac{K_0 \left(\frac{|z-t|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y} dt$$

combine

$$p_{Z+Q}(z) = p_Z * p_Q(z)$$

and

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“The sum of two random variables is equivalent to the multiplication of its moment-generating functions.

— no astronomer, ever

moment-generating function:

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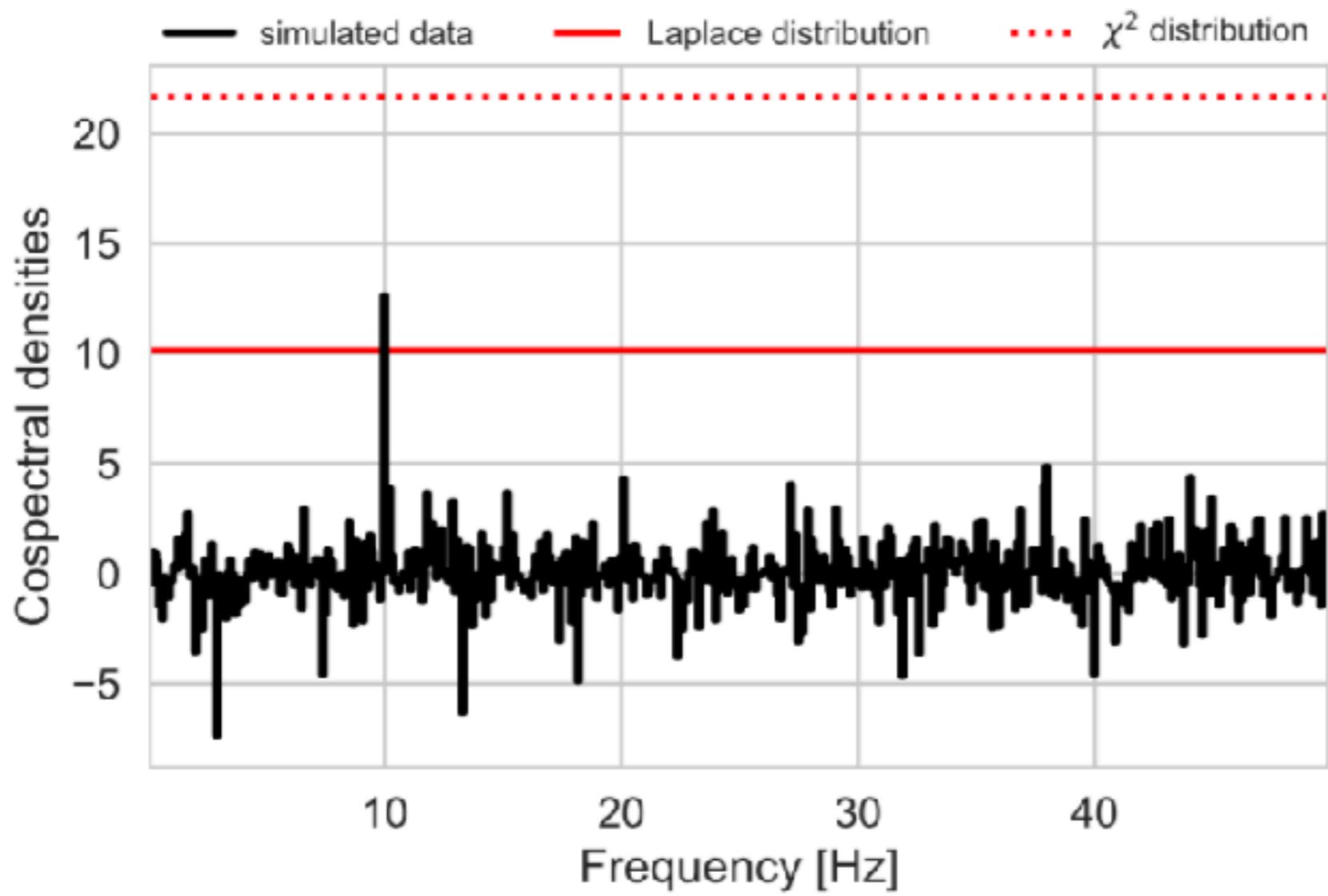
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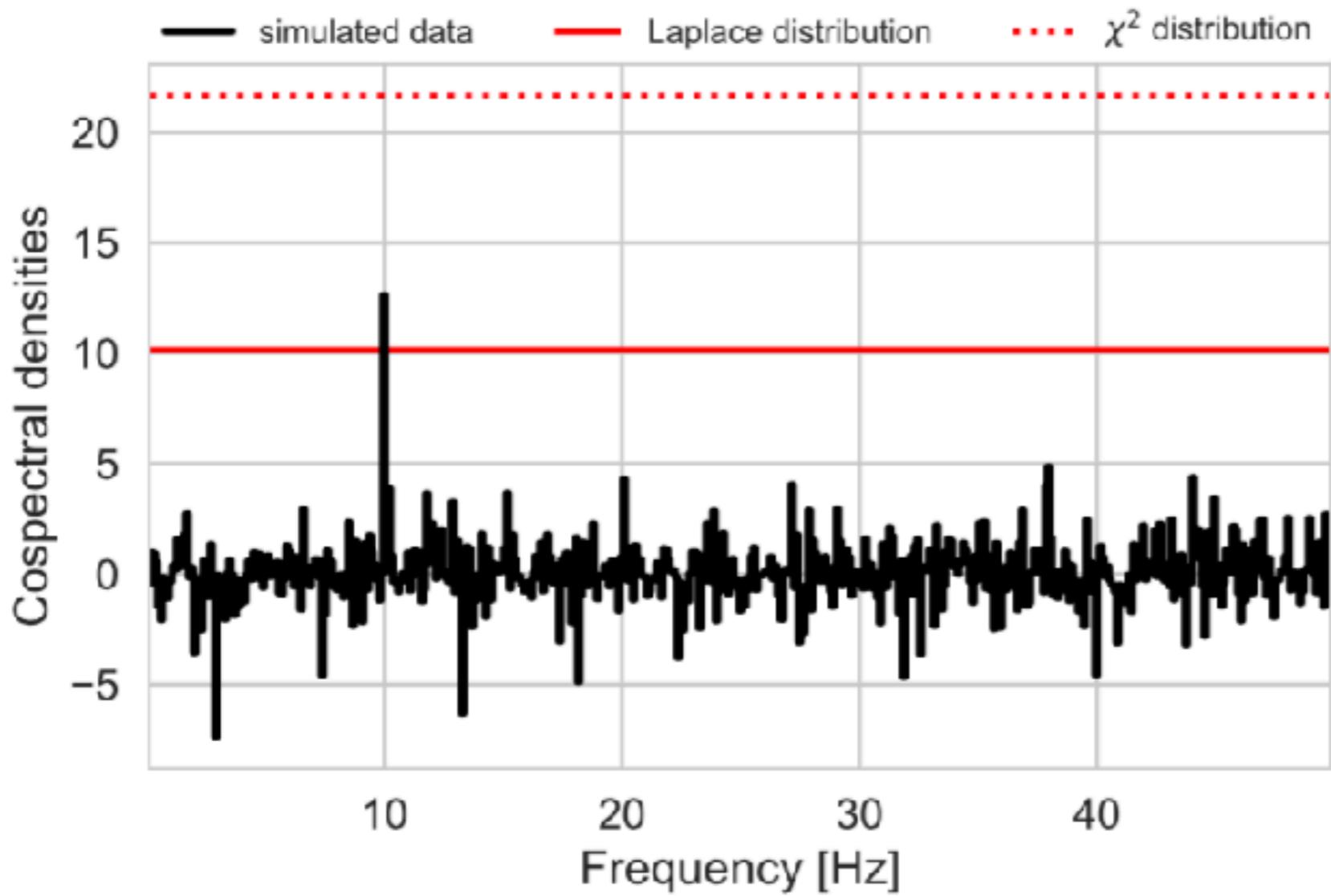
Seijas-Macías & Oliveira (2012)

This is a Laplace
distribution!

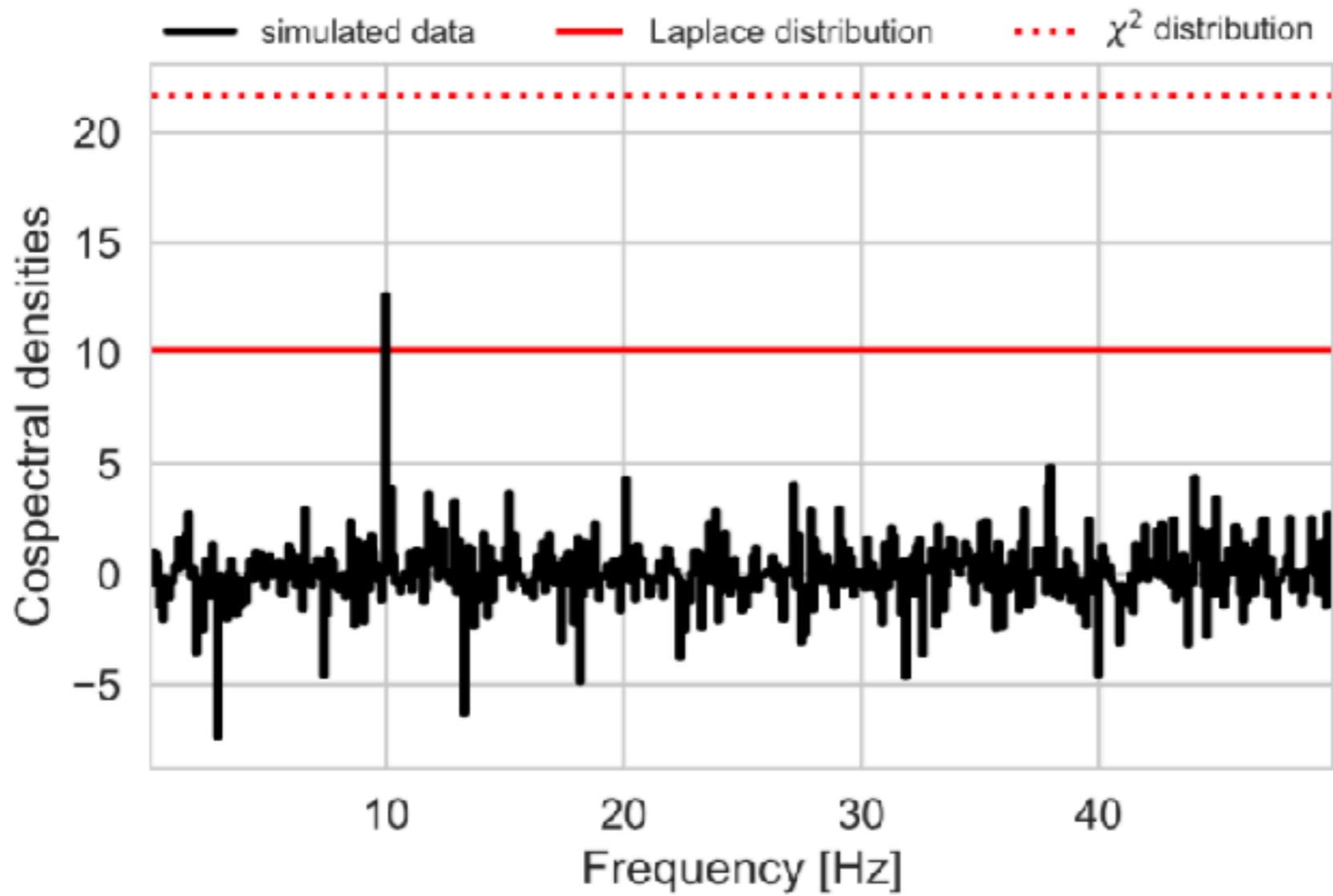
$$p(C_j|0, \sigma_x\sigma_y) = \frac{1}{\sigma_x\sigma_y} \exp\left(-\frac{|C_j|}{\sigma_x\sigma_y}\right)$$

... *why* are we doing this again?



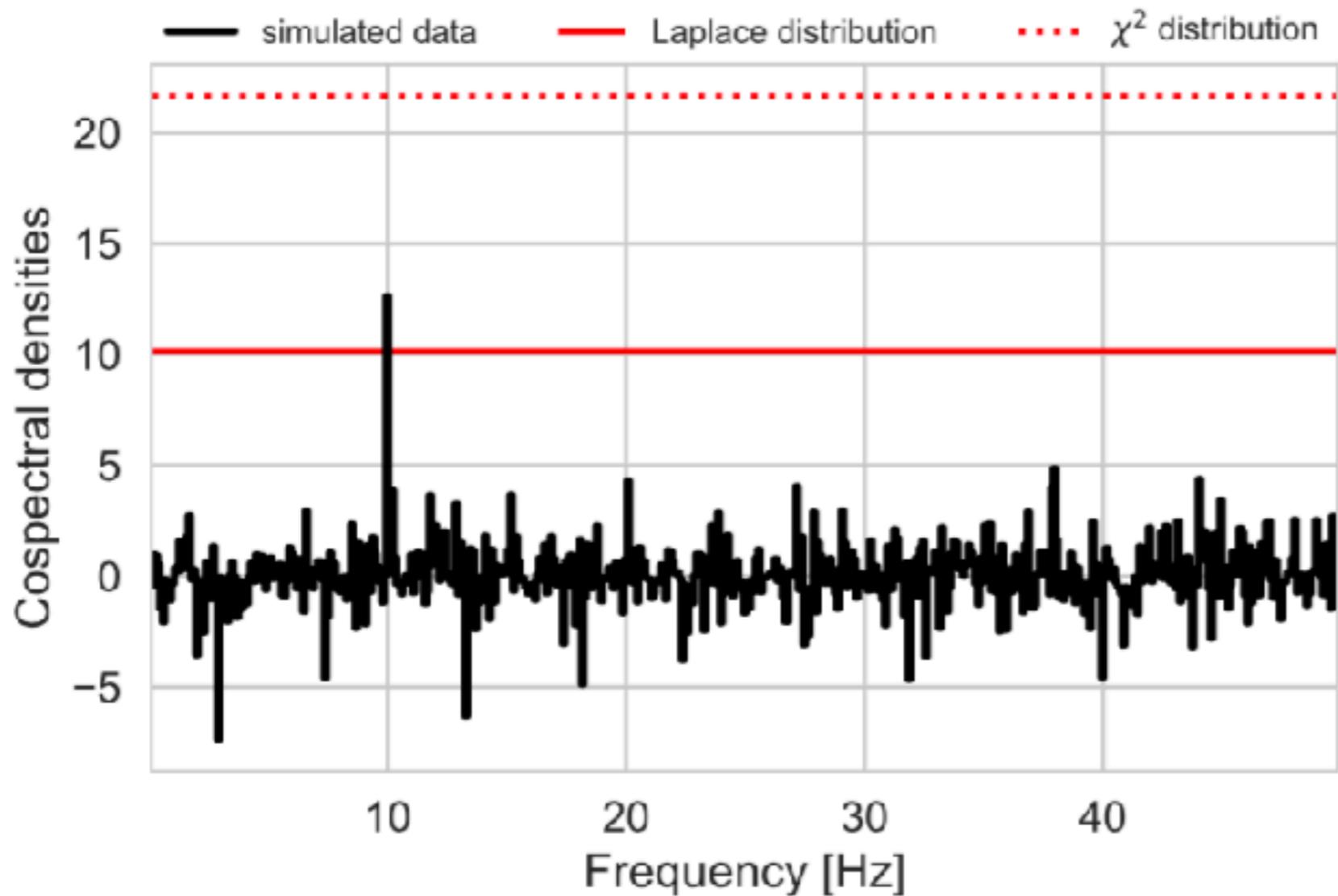


χ^2 distribution



χ^2 distribution

Laplace
distribution



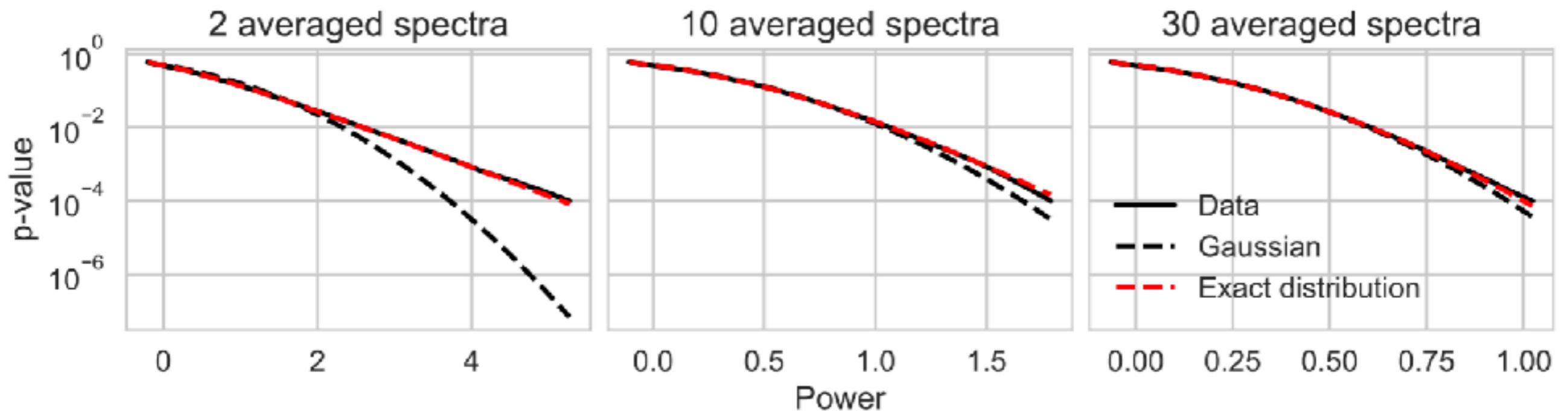
χ^2 distribution

Laplace
distribution

significance threshold matters!

what about averaged cospectra?

$$f_{\overline{X}_n}(x) = \frac{ne^{-|nx|}}{(n-1)!2^n} \sum_{j=0}^{n-1} \frac{(n-1+j)!}{(n-1-j)!j!} \frac{|nx|^{n-1-j}}{2^j}, x \in R.$$



**Gaussian approximation
works for large N**

so now we're done, right?

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... not quite!

- 1) Equations so far only work for white noise
- 2) The cospectrum only fixes the mean in the dead time case, not the variance!

**What about stochastic
variability?**

What about stochastic variability?

$$M_{C_j}(t) = \frac{1}{[1 - (1 + r)t][1 + (1 - r)t]}.$$

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correlation coefficient
between Fourier amplitudes

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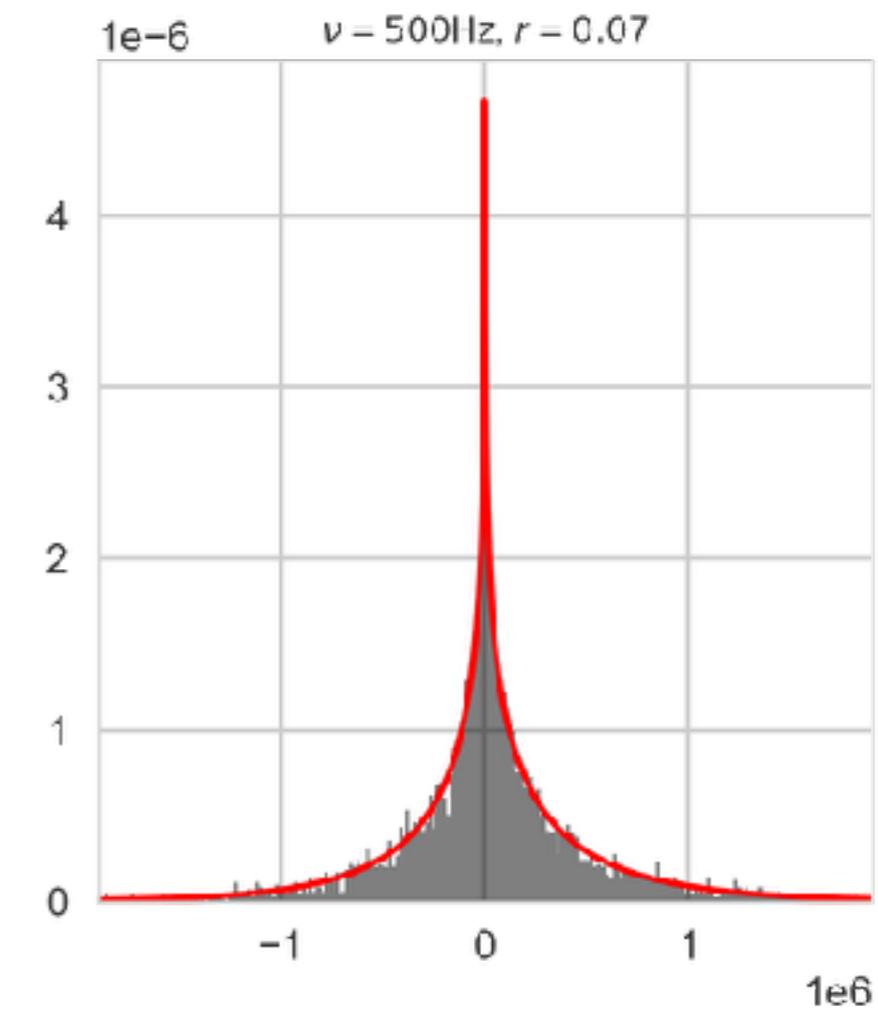
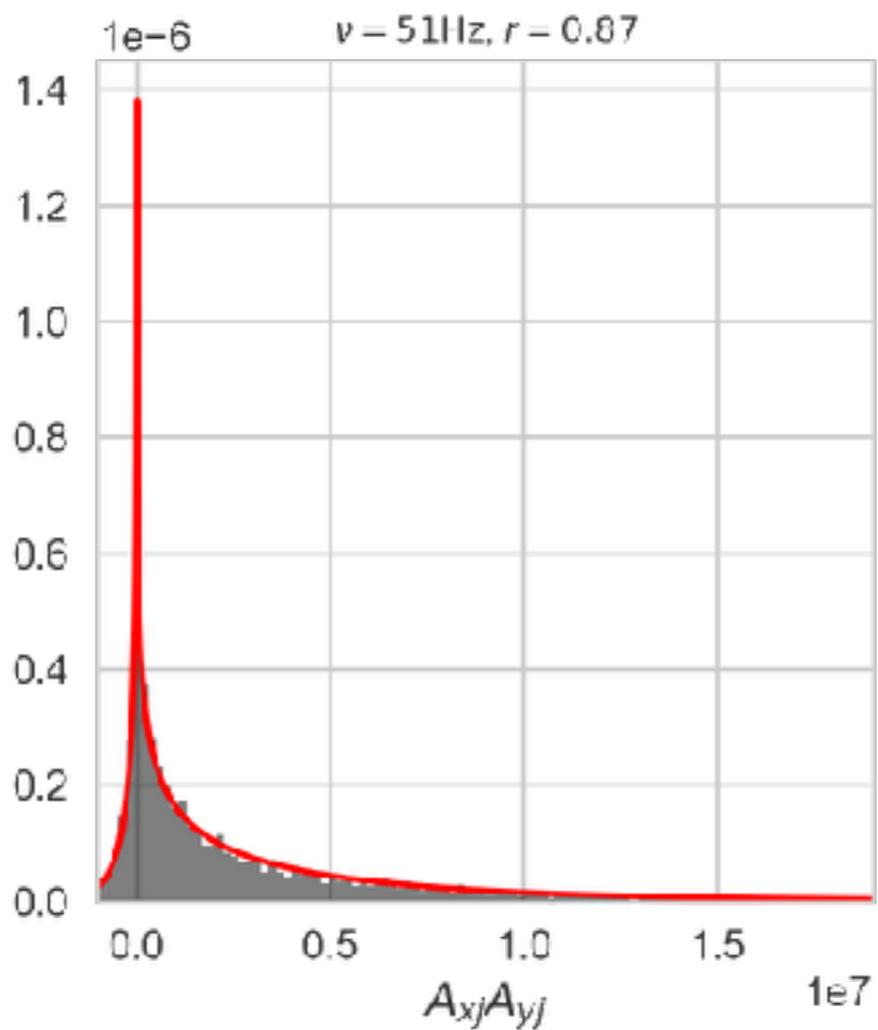
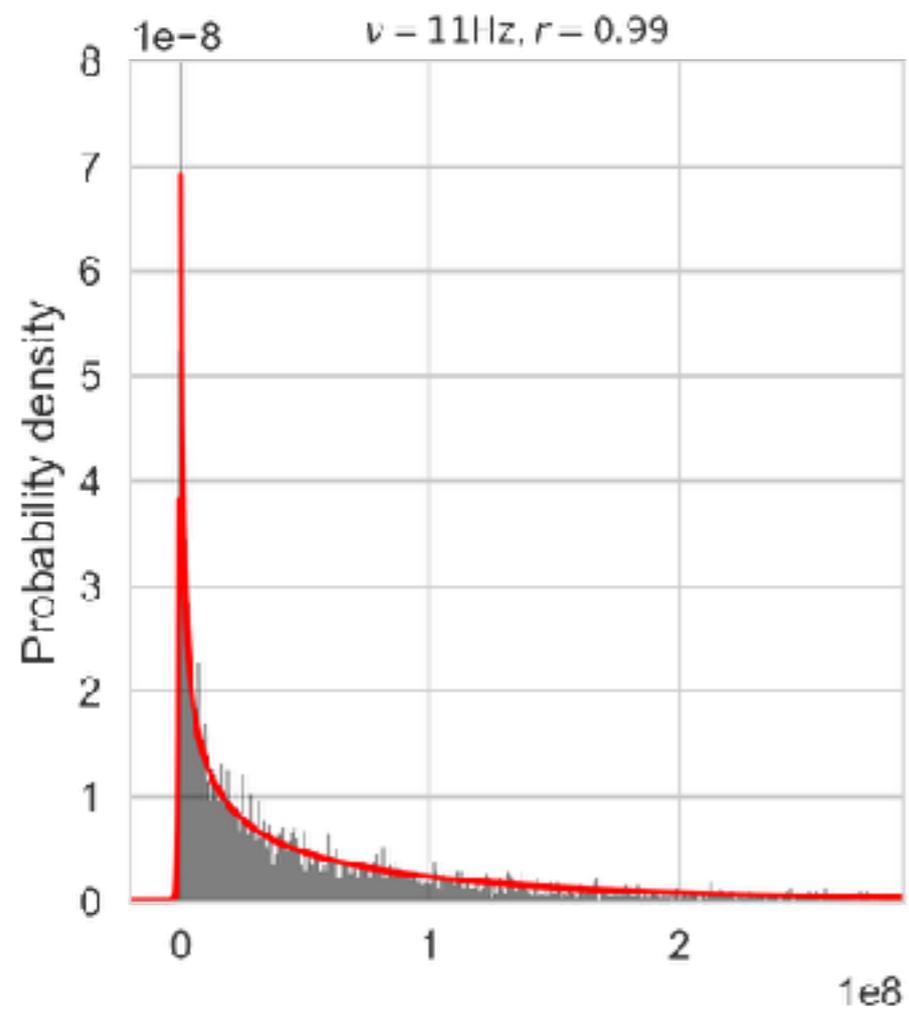
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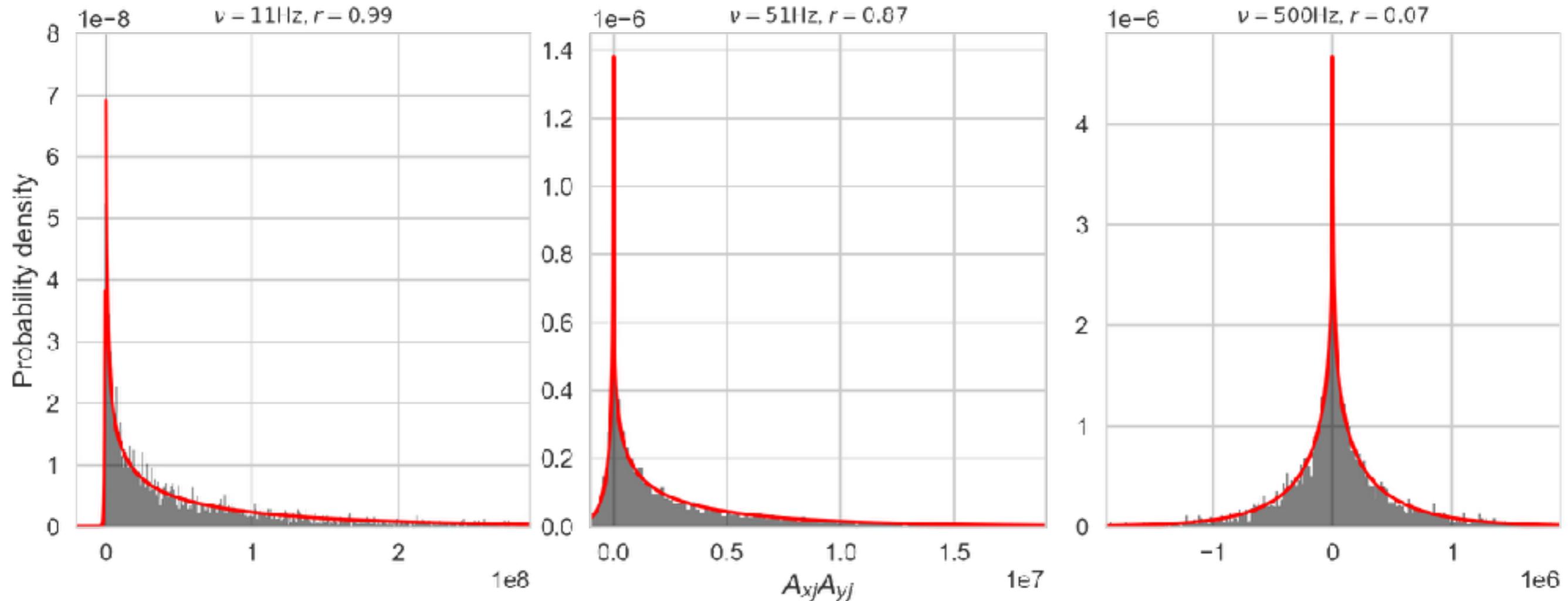
correlation coefficient
between Fourier amplitudes

$r = 0$ for white noise

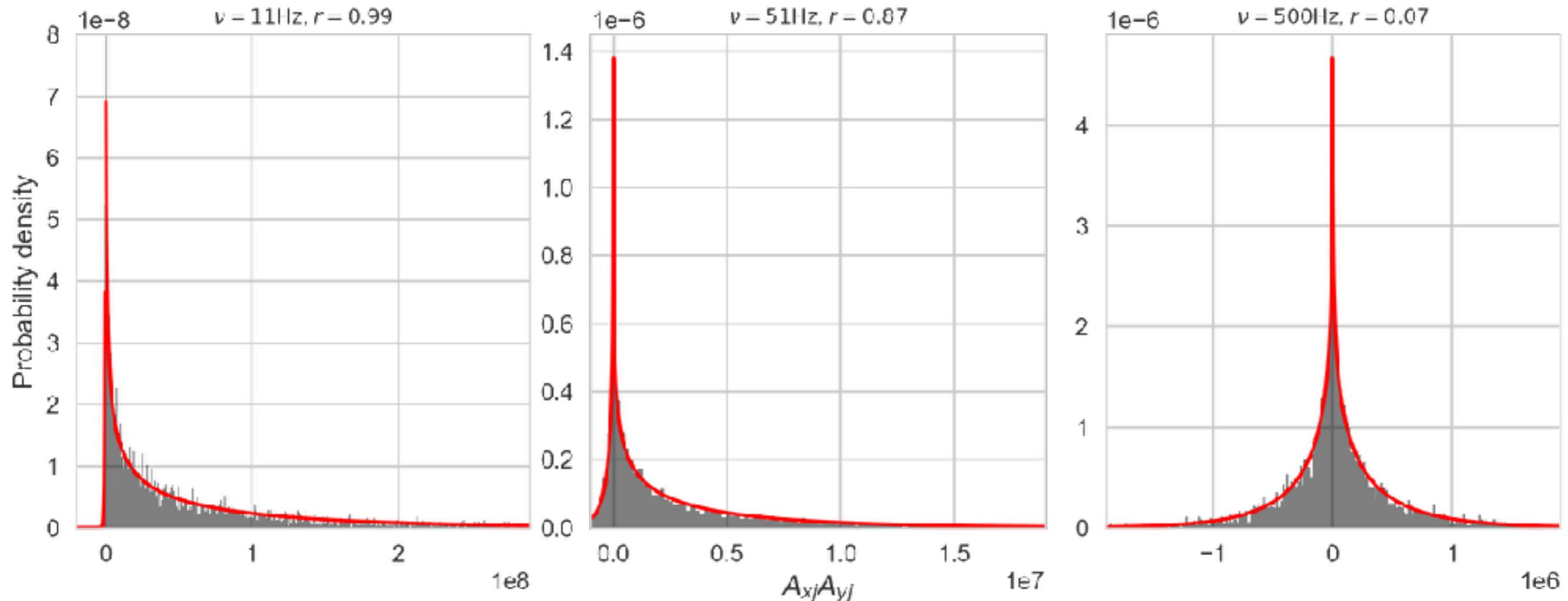
$r = 1$ for power spectra



statistical distribution depends on r ,
which depends on **power spectral shape!**

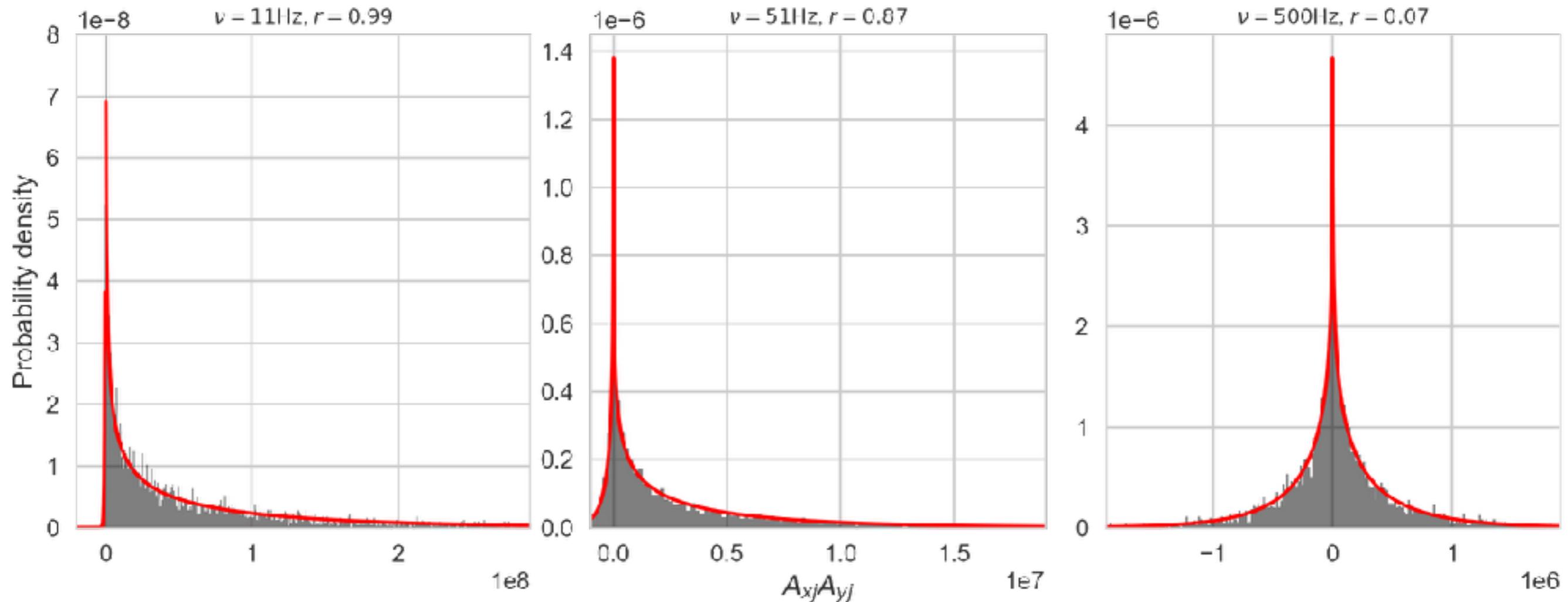


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also, there is no existing closed-form
solution for the PDF

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**Can we model the Fourier
amplitudes directly?**

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$$p(\{A_{xj}, A_{yj}, B_{xj}, B_{yj}\}_{j=0}^{N/2} | \theta, \lambda_{\text{phot}}) = \sum_{j=0}^{N/2} [-\log(2\pi|C|) - ([A_{xj}, A_{yj}]^T C^{-1} [A_{xj}, A_{yj}]) - ([B_{xj}, B_{yj}]^T C^{-1} [B_{xj}, B_{yj}])]$$

$$C = \begin{bmatrix} \sigma_{sj} + \sigma_n & \sigma_{sj} \\ \sigma_{sj} & \sigma_{sj} + \sigma_n \end{bmatrix}$$

Can we model the Fourier amplitudes directly?

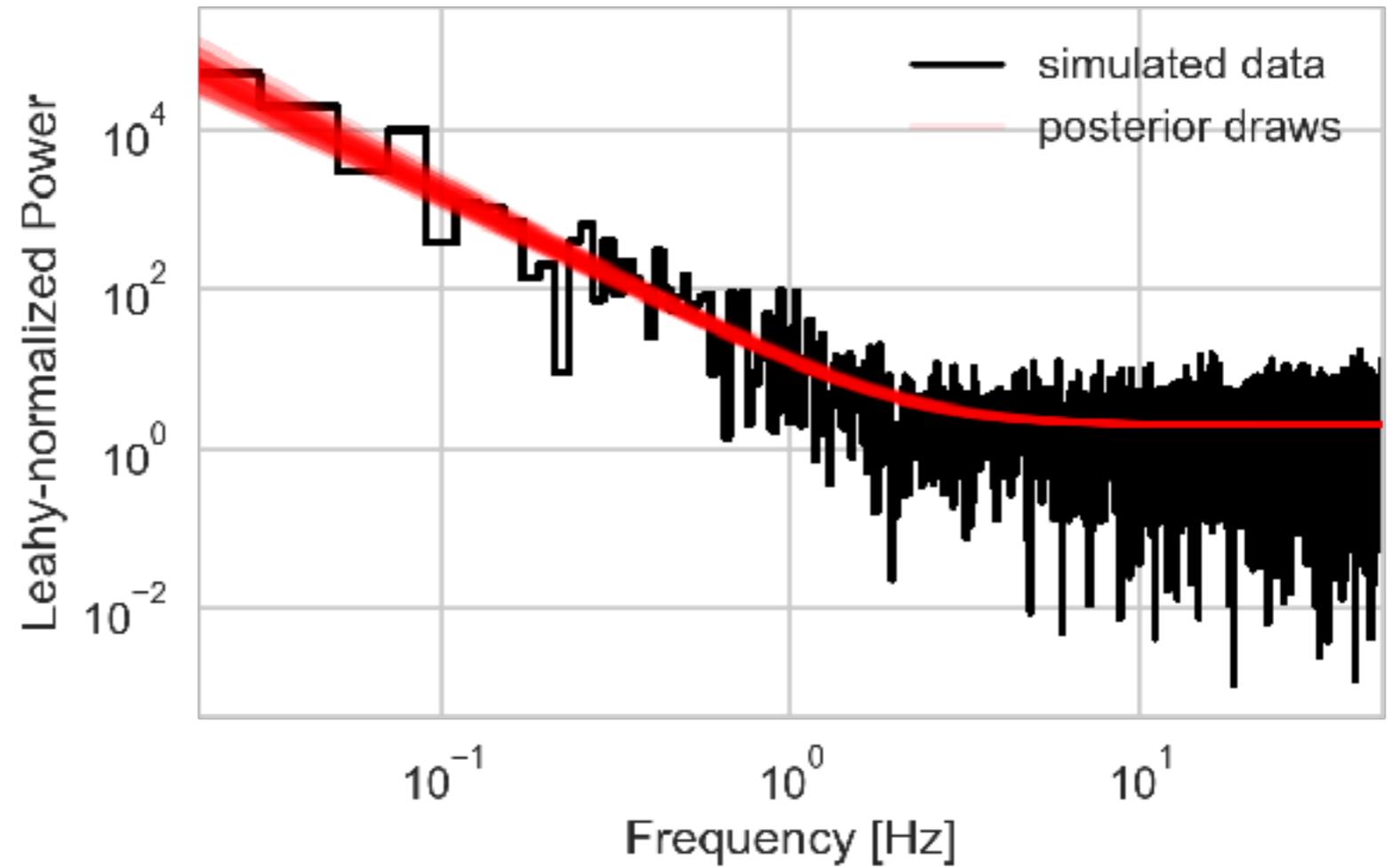
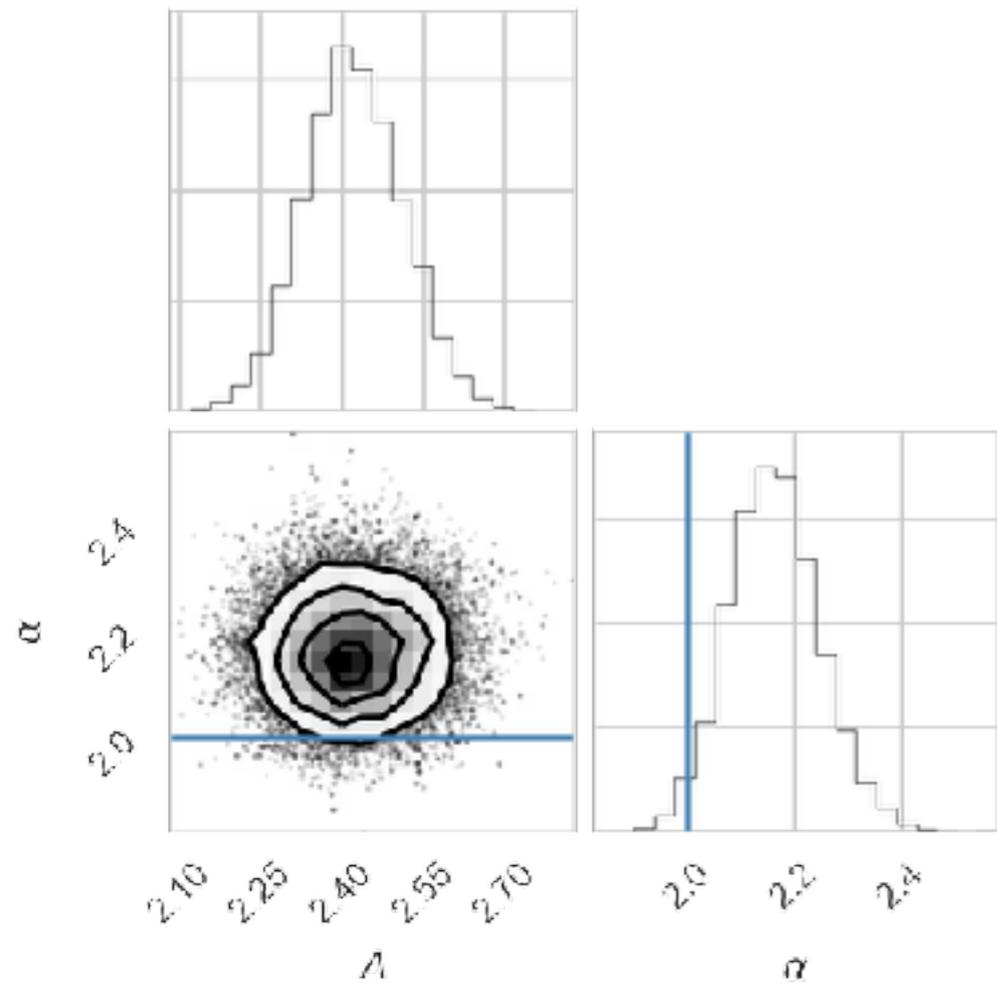
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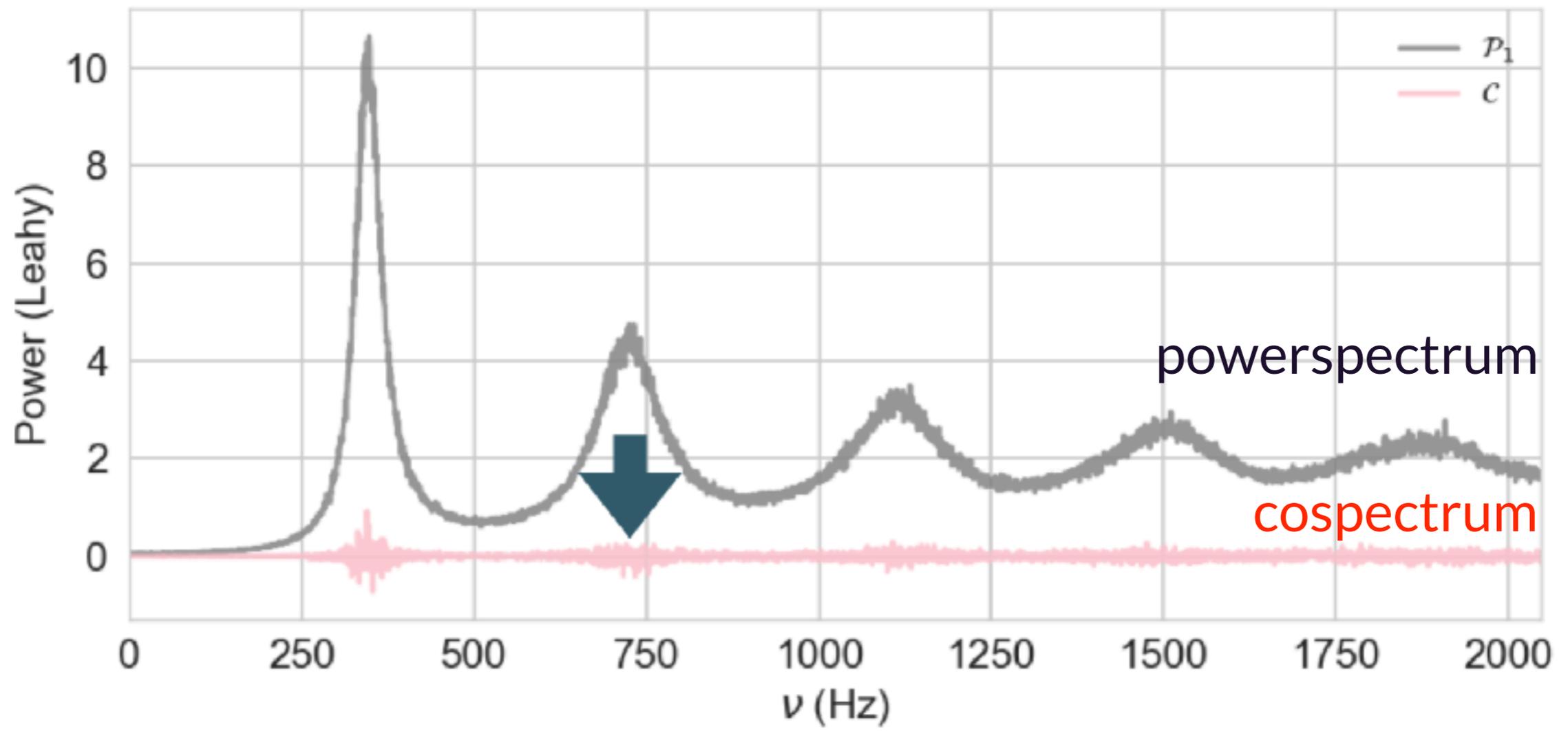
$$C = \begin{bmatrix} \sigma_{sj} + \sigma_n & \sigma_{sj} \\ \sigma_{sj} & \sigma_{sj} + \sigma_n \end{bmatrix}$$



depends on $P(\mathbf{v})$

[work in progress!]

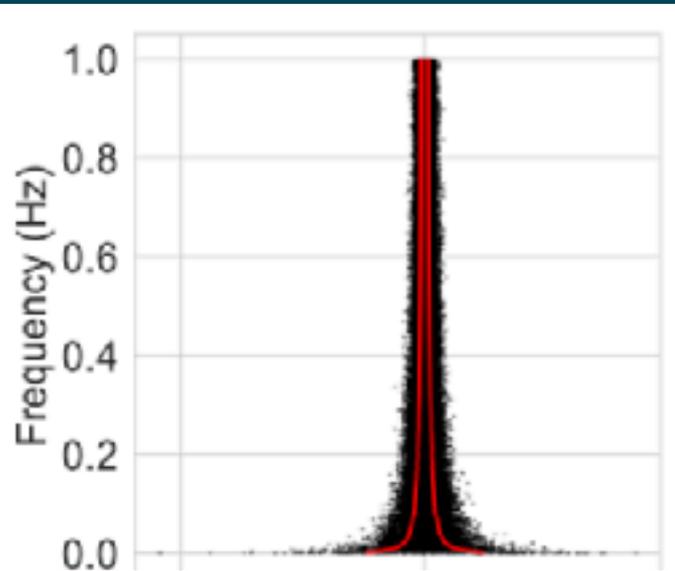




But: we can **correct** the
periodogram (and the
cospectrum) in some
cases!

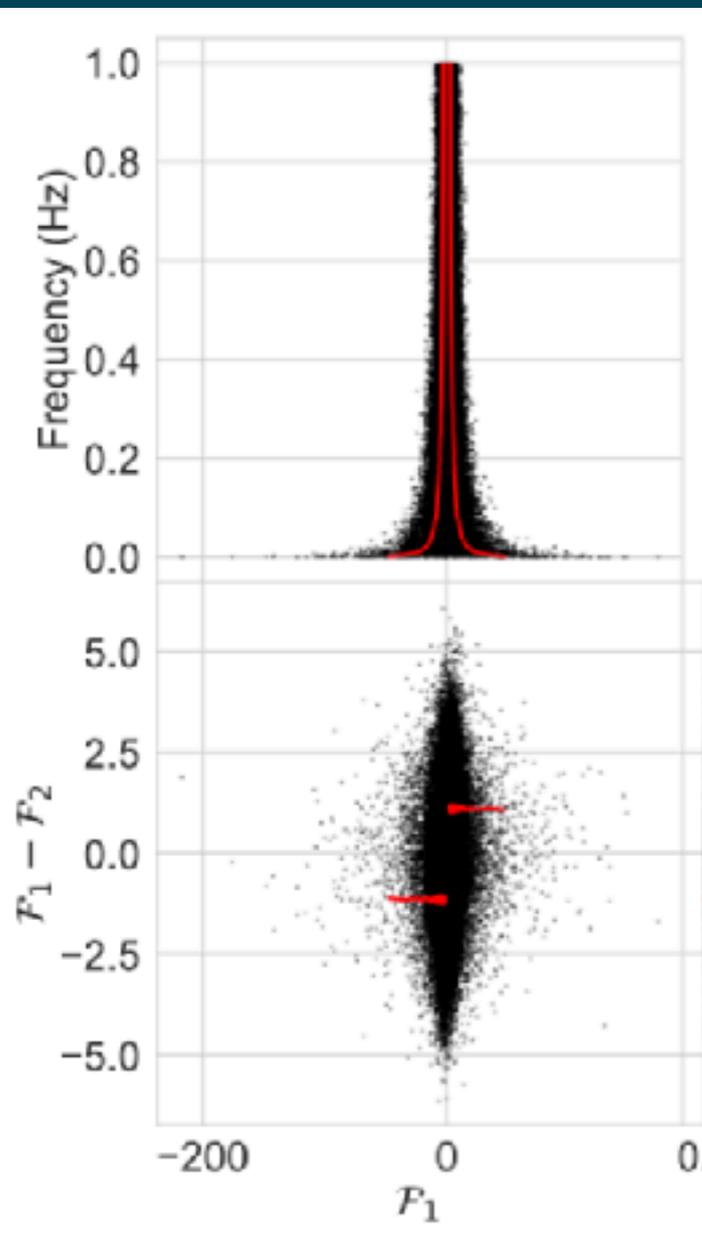
Fourier Amplitude Difference Correction

red noise, no dead time



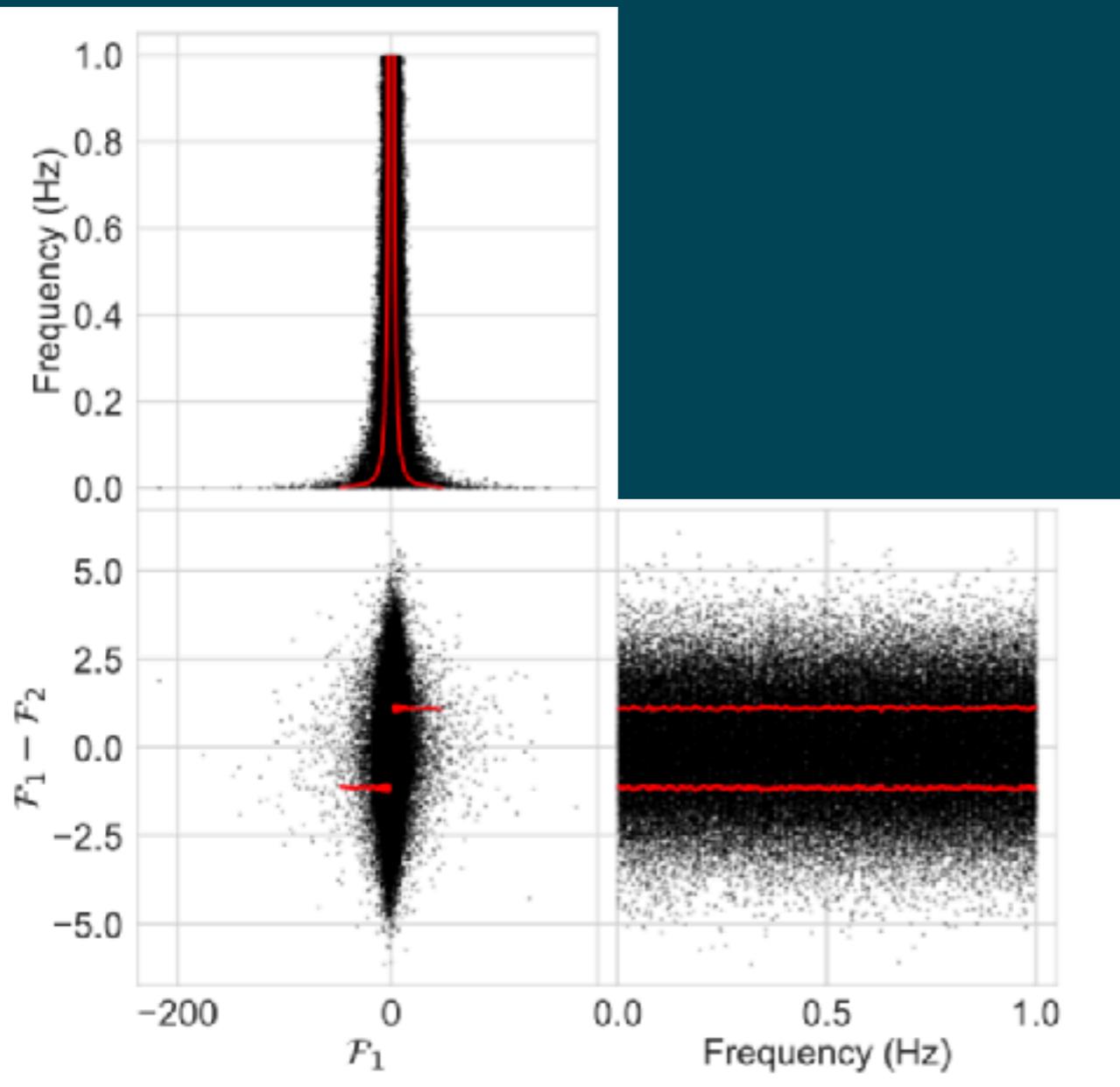
Fourier Amplitude Difference Correction

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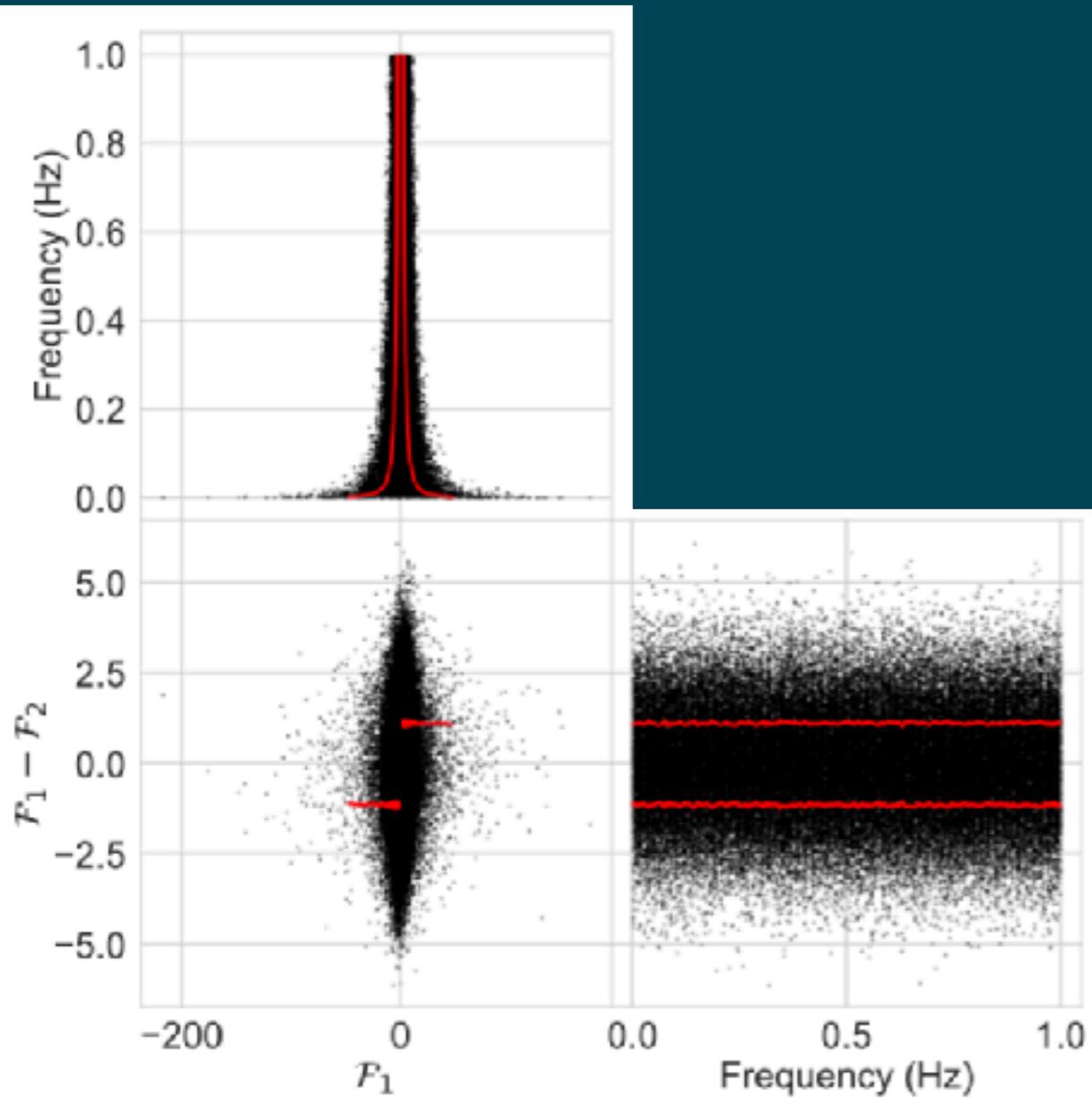
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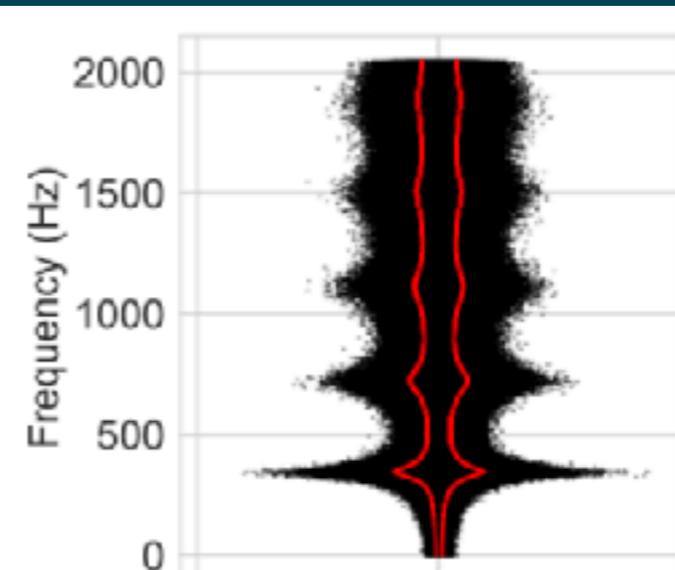


Fourier Amplitude Difference Correction

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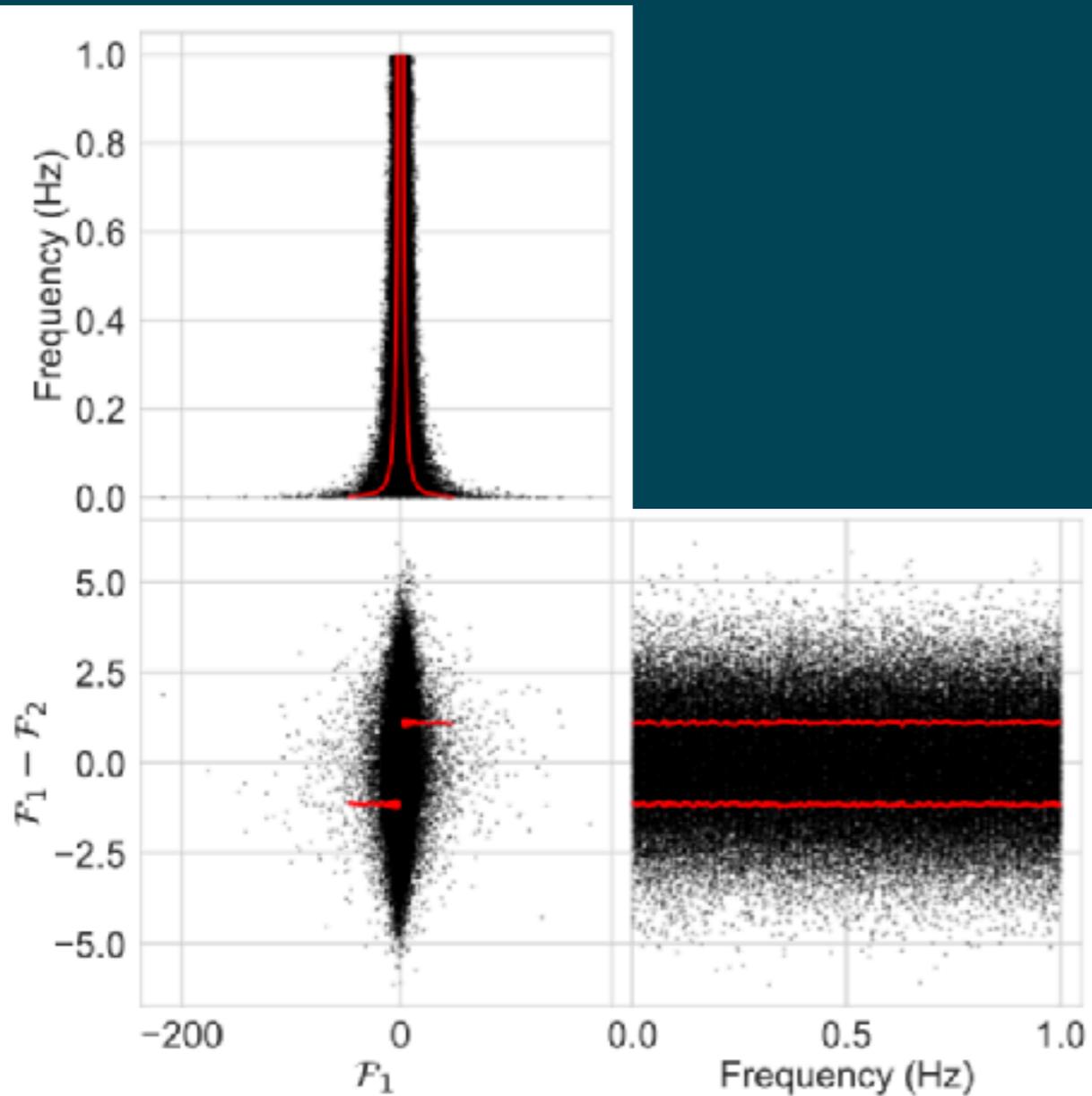


white noise, dead time

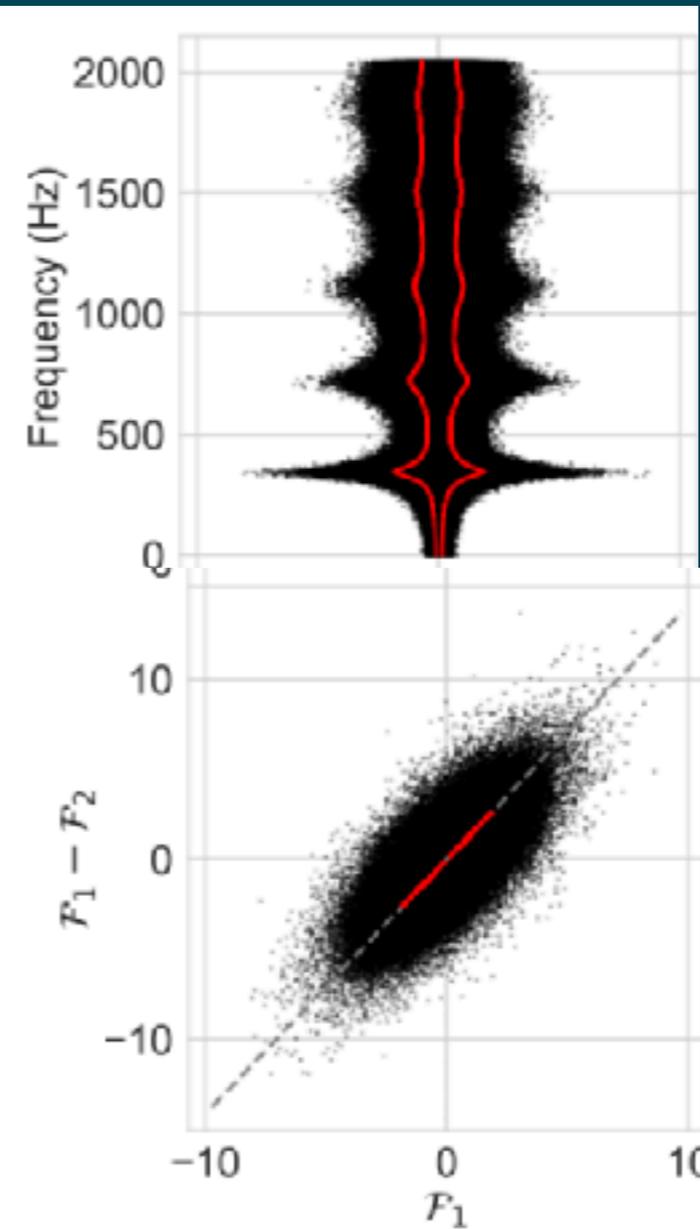


Fourier Amplitude Difference Correction

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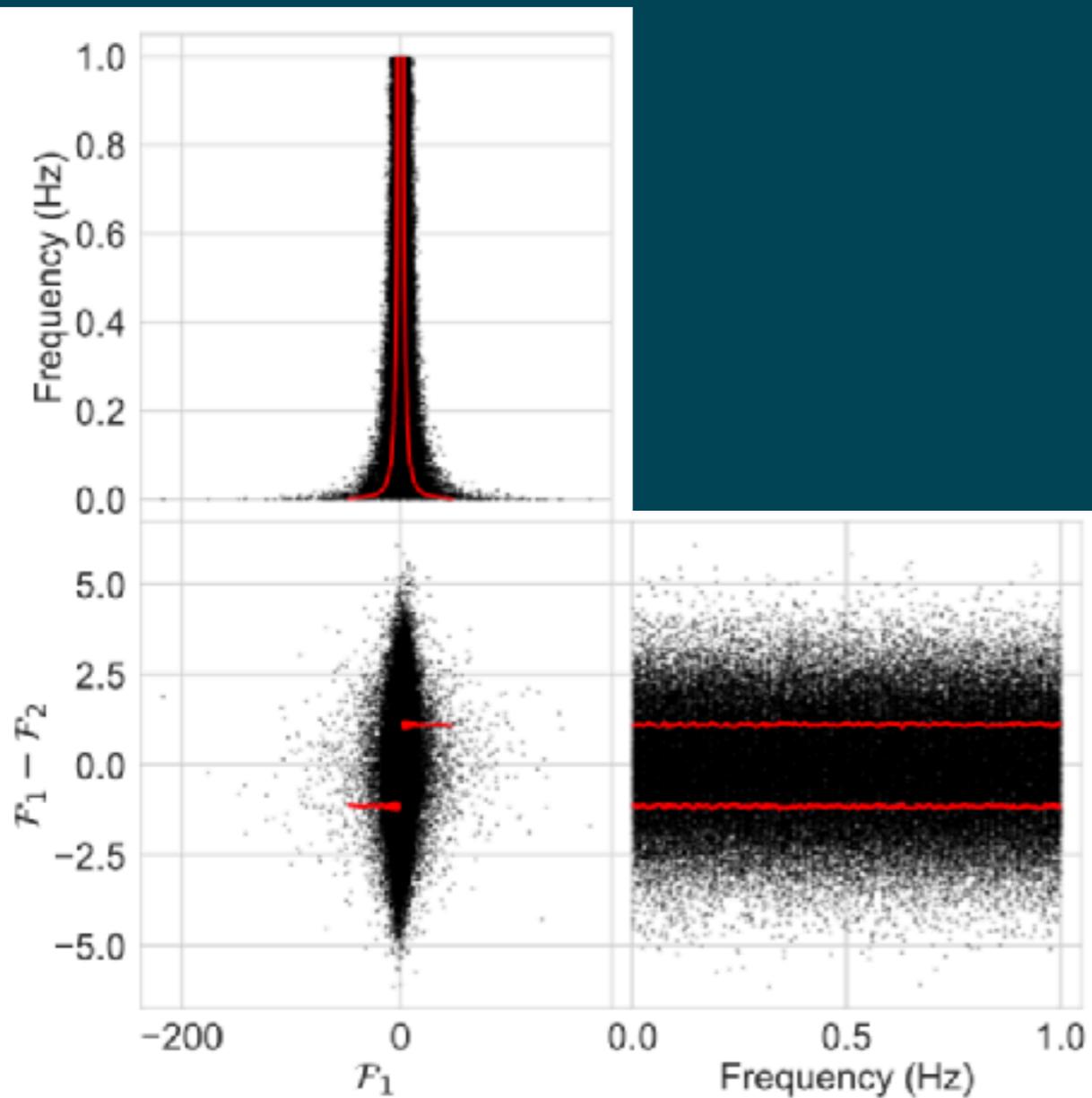


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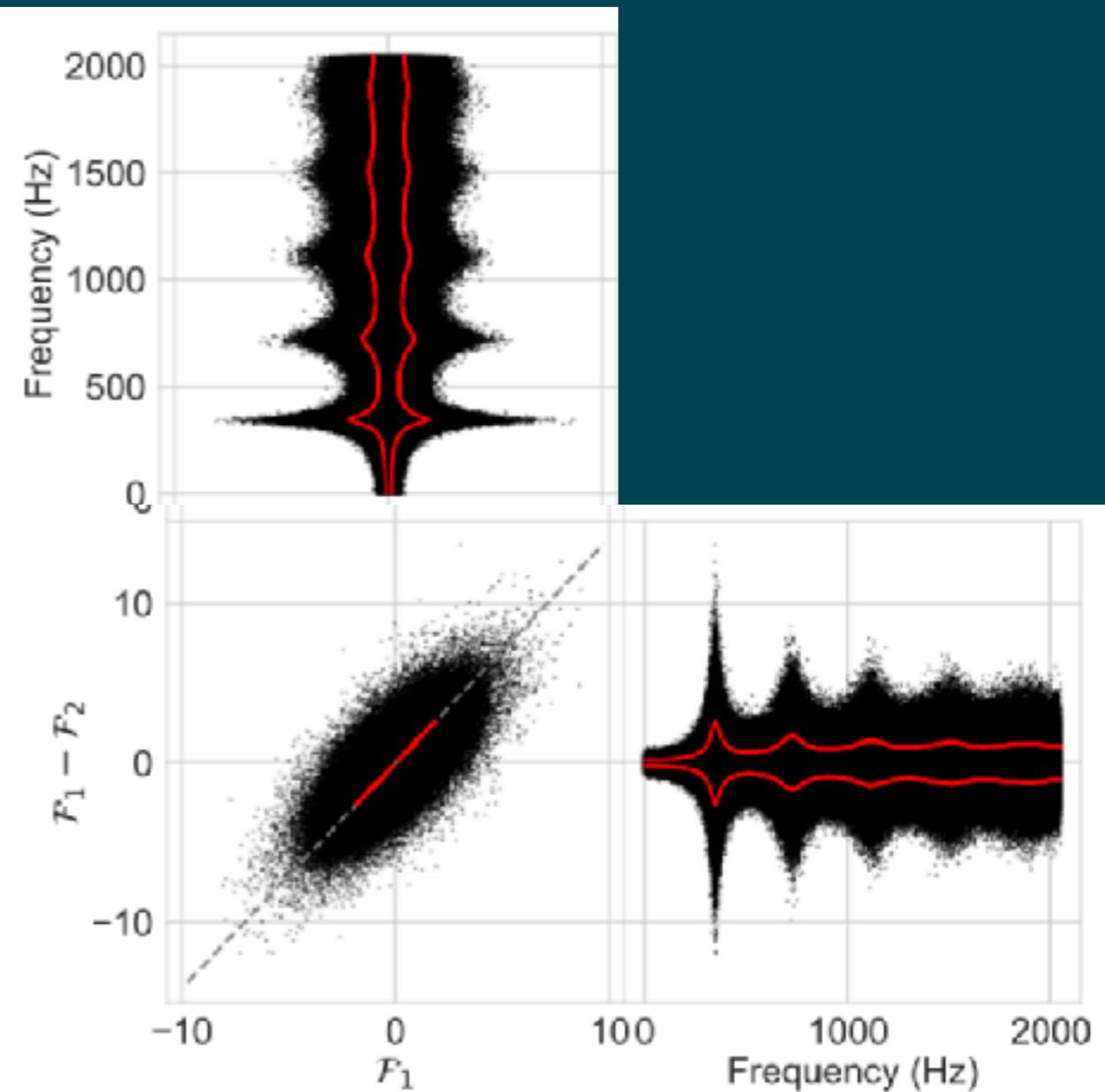


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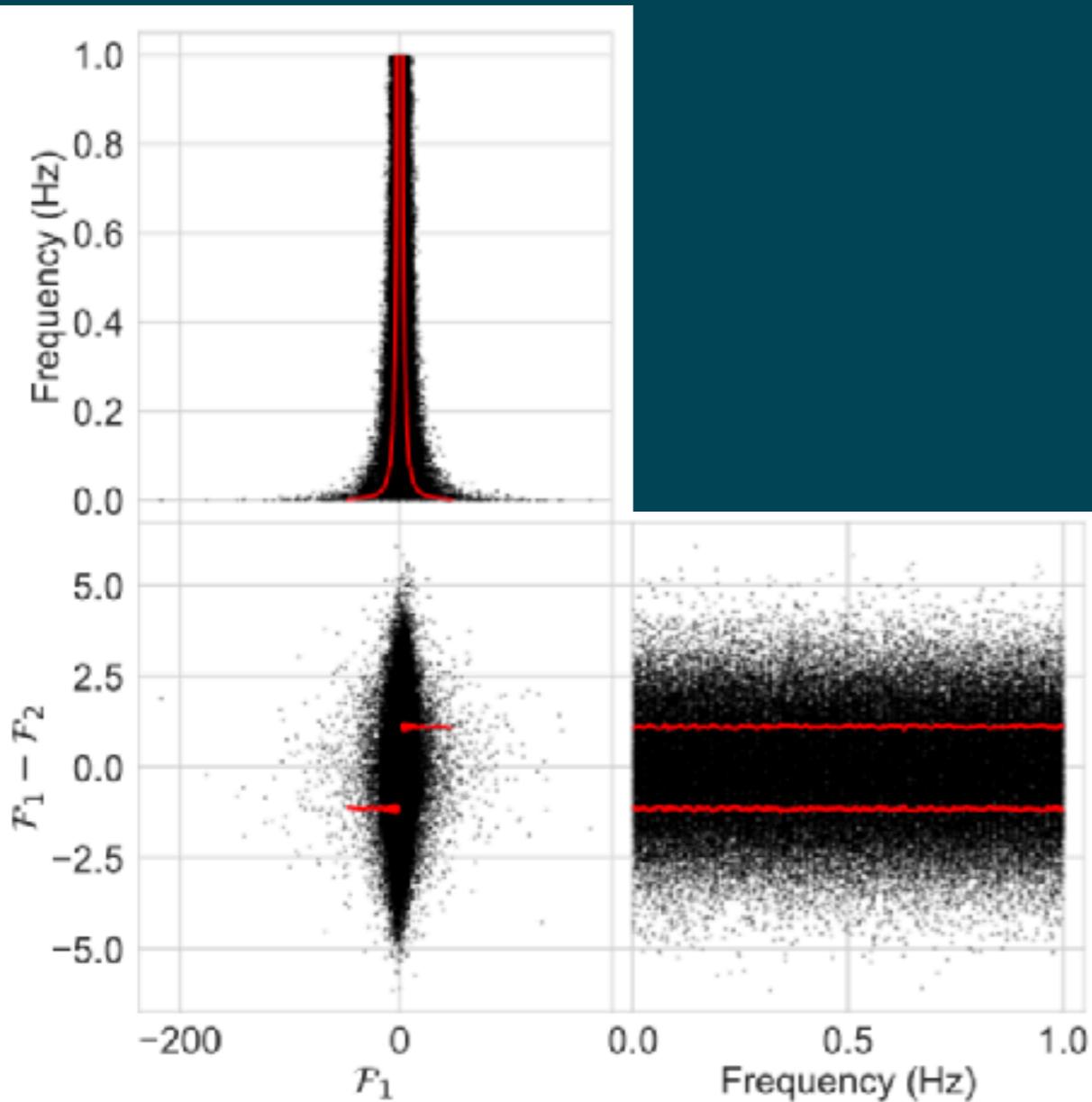


white noise, dead time

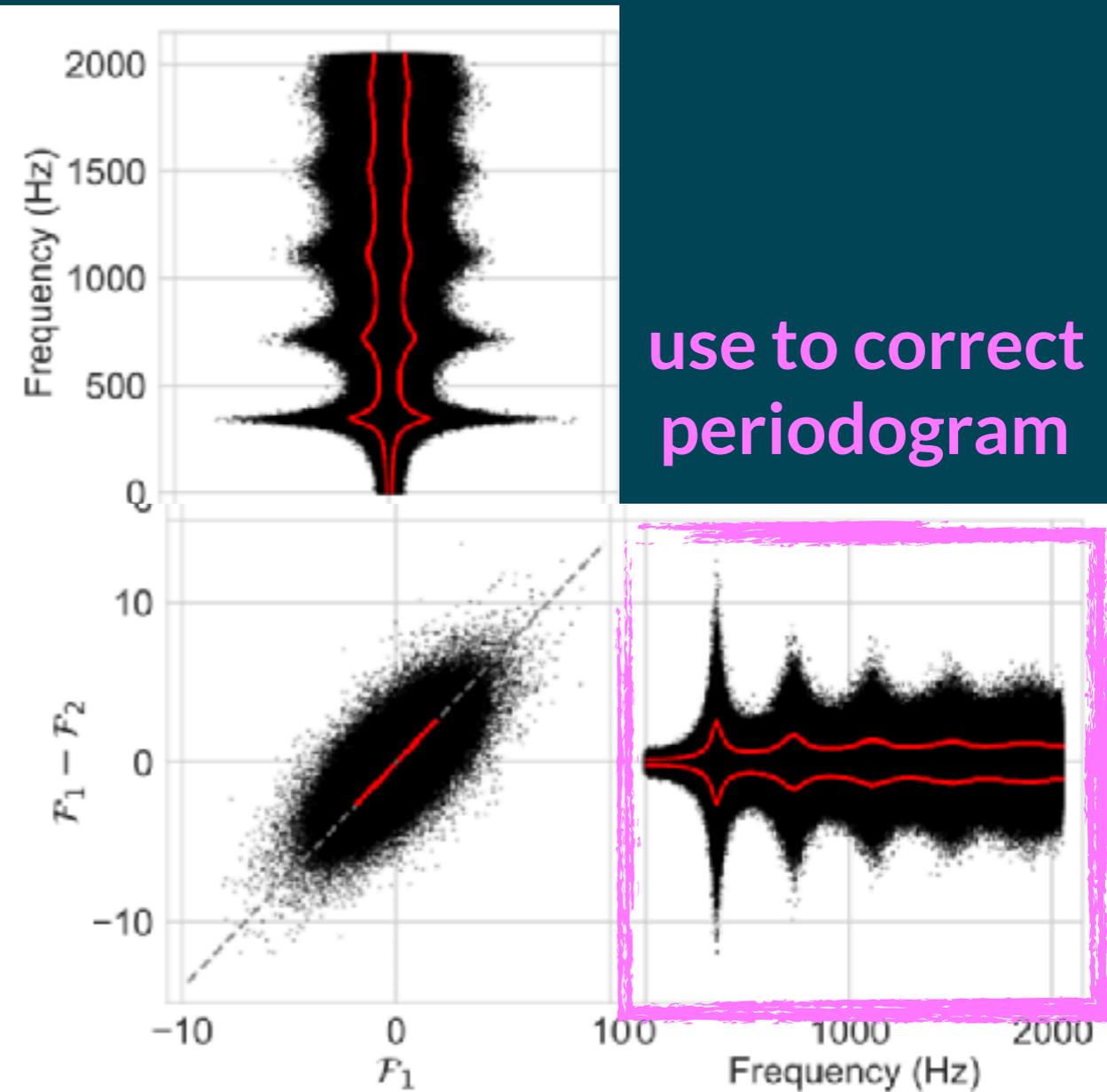


Fourier Amplitude Difference Correction

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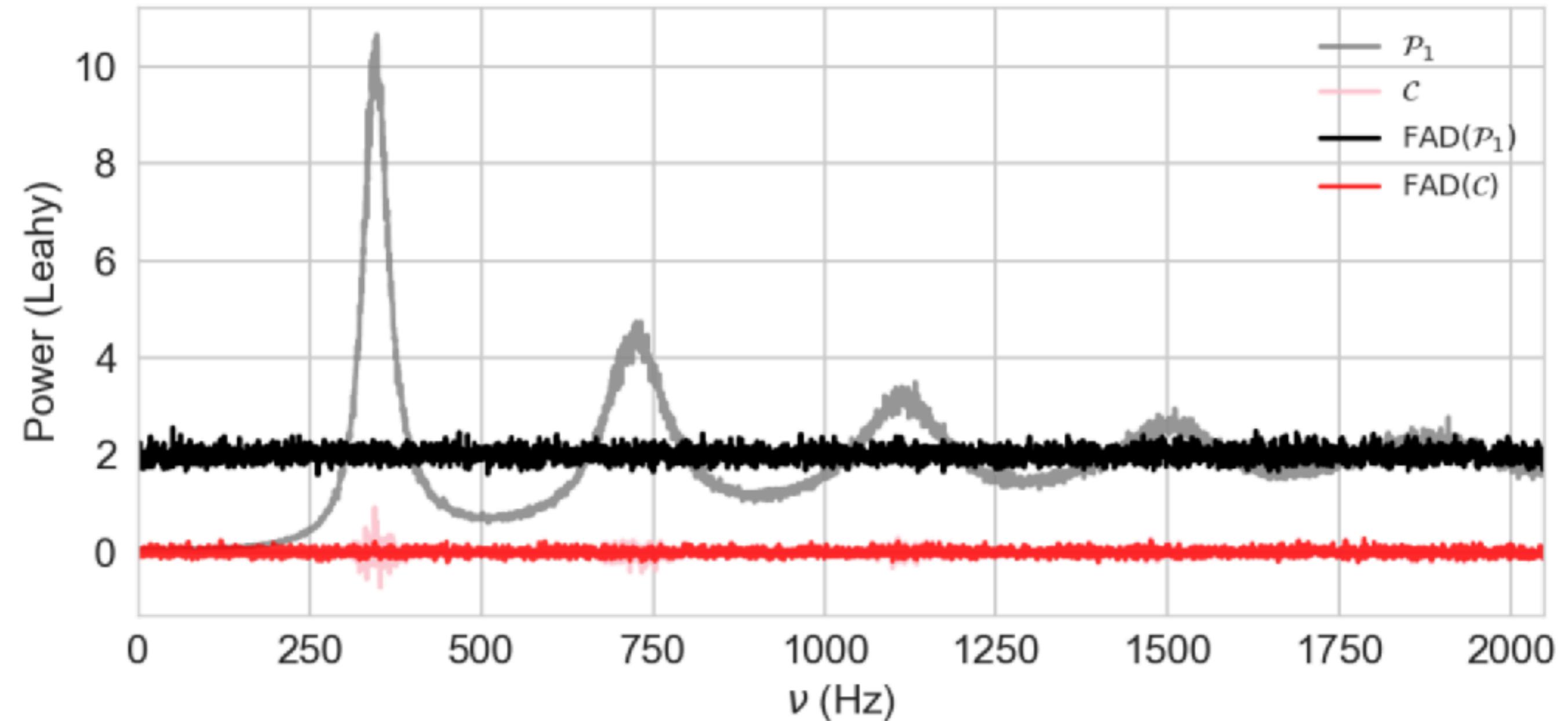


white noise, dead time



use to correct
periodogram

Fourier Amplitude Difference Correction



Caveat: this overestimates
the rms amplitude when
both flux and rms are very
large

Conclusions

- statistics with Fourier spectra is **fun!**
- use the **cospectrum** to do timing of **bright sources** in the presence of **dead time** when more than one detector is available (Bachetti+, 2015)
- the (averaged) cospectrum requires **different statistical distributions** for significance testing (Huppenkothen+Bachetti, arXiv:1709.09666)
- there is currently **no closed-form solution** for red noise cospectra (future work)
- but red noise periodograms can be **corrected** using the **FAD technique** (Bachetti+Huppenkothen, arXiv:1709.09700)



ASTRO HACK WEEK 2018

AUG 6 — AUG 10, 2018

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Google
Summer of Code

<h3>Python API for XSPEC</h3> <p>Develop a modular python API to use XSPEC (a popular X-ray astronomy tool) in python workflows</p>	<h3>Optimize Stingray for Large Datasets</h3> <p>Optimize tools in the Stingray library for use on large datasets from new X-ray space missions</p>	<h3>Phase-resolved oscillations</h3> <p>Implement method to calculate the phase of oscillatory phenomena with non-constant frequency, and calculate phase-resolved spectra</p>
<p>dhuppenkothen abigailStev</p>	<p>matteobachetti pbalm abigailStev</p>	<p>abigailStev matteobachetti</p>
<p>GSOC</p>	<p>GSOC</p>	<p>GSOC</p>
<p>timelab</p>	<p>timelab</p>	<p>timelab</p>

<http://openastronomy.org/gsoc/gsoc2018/>