Astro 193 : Feb 18

- Follow up
  - projects
- MI dof -- yes, the square is outside
  \[ \text{dof} = (N-1) \left( 1 + \frac{N}{(N+1)} \right) \cdot \left( \frac{W_{ii}}{B_{ii}} \right)^2 \]
- least-squares fitting and HW 4
- Moments and intervals
- Random Numbers
- Binomial, Poisson, Gamma, Normal, $\chi^2$, Student’s t
Moments

- \( M_m = \int x^m p(x) \, dx \) or \( M_m = \sum x_i^m p(x_i) \)

\[ M_0 = 1, \text{ mean} = M_1, \text{ variance} = M_2 - M_1^2 \]

- Moment Generating Function

\[ e^{\xi x} f(x) \rightarrow M_m = \frac{\partial^n M}{\partial x^n} \]

- Characteristic Function

\[ \varphi(\nu) = \int e^{ivx} f(x) \, dx \equiv \sum (iv)^k / k! \, M_k \]

\[ f(x) = \int dv \, e^{-ivx} \sum (iv)^k / k! \, M_k \]
Intervals

- \([a,b]\) such that \(\int_{[a,b]} p(x) \, dx = c\)
  - non-unique
  - equal-tail intervals
  - highest-posterior density intervals
- \(1\sigma\) - one-sided and two-sided
  - Gaussian-equivalent \(1\sigma\)
Random Numbers
• What are random numbers?
• PRNGs — Pseudo Random Number Generators
• Random draws from distributions
• Rejection sampling
Random Number

• How many bits are needed to represent the number?

• Entropy: $H(X) = \sum_i p(X_i=x) \log_2 p(X_i=x)$

• Practical issue: must represent a known distribution with high fidelity
PRNG

- Generate long, seemingly random sequences that are fully deterministic, keyed to a short initial seed

- If internal state has $n$ bits, repeats with period at most $2^n$

- Beware of: short periodicity, correlations, weak seeds, lack of uniformity, failure against Zipf-law like tests
PRNG-Examples

- digits of $\pi$, $\sqrt{2}$, etc.
- middle-square method (von Neumann): take a digit, square it, pick the middle digits
- multiply-with-carry: $(ax_k + b_k) \mod 2^{32}$, $(ax_k + b_k)/2^{32}$
- Mersenne twister: MT19937 — default in R, python, IDL, Matlab, Ruby, PHP, etc. Not cryptographically secure, but optimized for Monte Carlo sampling
Draws from Distributions

- cdf method: from analytical or tabulated 1D functions
  - trivial example: $\delta$-function
  - $\delta$-function with a pedestal
  - arbitrary cdf: given $r=F(x)$, compute $x=F^{-1}(r)$
Rejection sampling

- Suppose we cannot sample $f(\theta)$ directly, but can find $g(\theta)$ such that $f(\theta) \leq M \cdot g(\theta)$

- sample $\theta' \sim g(\theta)$

- sample $u \sim \text{unif}(0,1)$

- if $u \leq f(\theta')/Mg(\theta')$, accept $\theta'$, otherwise reject $\theta'$ and sample again
Rejection sampling

• Extra reading: Chapter 3 of Robinson
Distributions

Reading: Chapters 2 and 4.5 of Robinson
Chapter 4 of Eadie et al
Distributions

- What are distributions? where do they come from? how to use them?
- Binomial, Poisson, Normal, Student’s t, $\chi^2$
R> help(distributions)

For the beta distribution see dbeta.

For the binomial (including Bernoulli) distribution see dbinom.

For the Cauchy distribution see dcauchy.

For the chi-squared distribution see dchisq.

For the exponential distribution see dexp.

For the F distribution see df.

For the gamma distribution see dgamma.

For the geometric distribution see dgeom. (This is also a special case of the negative binomial.)

For the hypergeometric distribution see dhyper.

For the log-normal distribution see dlnorm.

For the multinomial distribution see dmultinom.

For the negative binomial distribution see dnbinom.

For the normal distribution see dnorm.

For the Poisson distribution see dpois.

For the Student's t distribution see dt.

For the uniform distribution see dunif.

For the Weibull distribution see dweibull.
Binomial

- One or the other, with probability $\rho$
- $k$ of one, out of a total of $N$
- $p(\rho; N, k) = \binom{N}{k} \rho^k (1-\rho)^{N-k}$
- cf. $(p+q)^N$
- $M_{m+1} = N\rho \; M_m + \rho(1-\rho) \; \partial M_m / \partial \rho$

\[ M_0=1, \; M_1=N\rho, \; M_2=N^2\rho^2 + N\rho(1-\rho), \; \text{variance} = N\rho(1-\rho) \]
Poisson

• $N \gg k$, $\rho = R\delta t/N$, $N\rho \equiv \mu$ finite

• $p(k|\mu) = (1/k!) \mu^k e^{-\mu}$

• $M_{m+1} = \mu M_m + \mu \partial M_m / \partial \mu$

• $M_0=1$, $M_1=\mu$, $M_2=\mu^2 + \mu$, variance $= \mu$