Dark Sources Detection

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Introduction: Data and Project Goal

- Data:
  - Y_i, observed photon counts, contaminated with background in a source exposure.
  - X, observed photon counts in the exposure of pure background .
- Goals of the Project:
  - To develop a fully Bayesian model to infer the distribution of the intensities of all the sources in a population
  - To identify the existence of dark sources in the population

A Brief Review: Bayesian Model

- Level I model:

$$
\begin{aligned}
X|\xi &\sim Pois(\xi), \\
Y_i = Y_{iB} + Y_{iS}&, \text{ where } Y_{iB}|\xi \sim Pois(a_i\xi), \\
Y_{iS}|\lambda_i \sim Pois(b_i\lambda_i) &\sim \begin{cases} \delta_0, & \text{if } \lambda_i = 0; \\ Pois(b_i\lambda_i), & \text{if } \lambda_i \neq 0. \end{cases}
\end{aligned}
$$

  - ξ is the background intensity,
  - λ_i is the intensity of source i,
  - a_i is ratio of source area to background area (known constant),
  - b_i is the telescope effective area (known constant).

A Brief Review: Bayesian Model

- Level II model:

$$\lambda_i|\alpha,\beta,\pi \begin{cases} = 0, & \text{with probability } 1-\pi; \\ \sim \mathit{Gamma}(\alpha,\beta), & \text{with probability } \pi. \end{cases}$$

- Level III model:

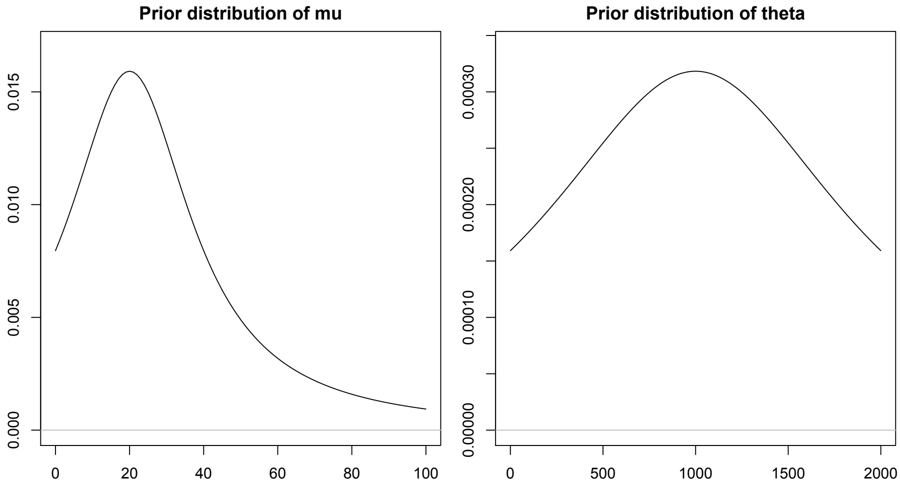
$$P(\alpha,\beta,\pi) \propto P(\alpha,\beta).$$

Weakly Informative Prior on α, β

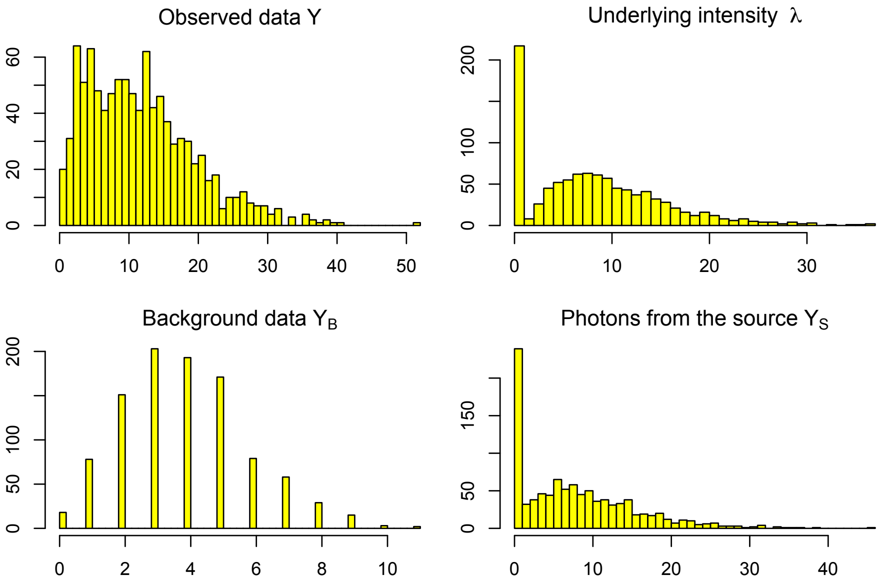
- The prior distribution of α, β needs to be proper
- We do not want the proper prior to be very informative
- Let $\mu = \dfrac{\alpha}{\beta}, \theta = \dfrac{\alpha}{\beta^2}$ be the mean and variance parameters of the Gamma distribution.
- Weakly informative prior on μ, θ:

$$P(\mu) \propto \frac{1}{1 + \left(\frac{\mu-20}{20}\right)^2}, \quad P(\theta) \propto \frac{1}{1 + \left(\frac{\theta-1000}{1000}\right)^2}$$

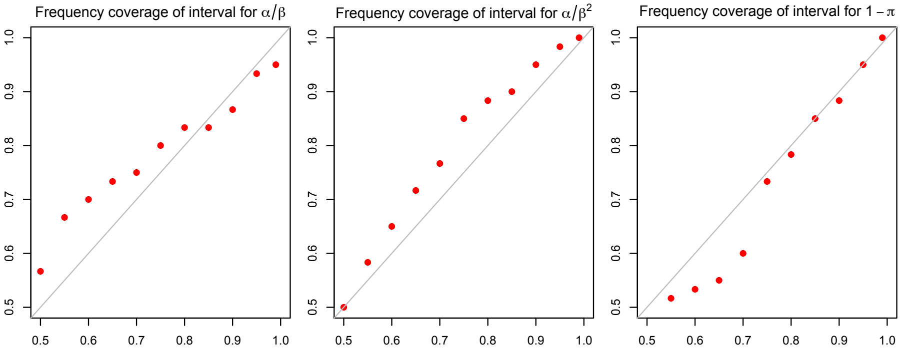
Weakly Informative Prior on α, β



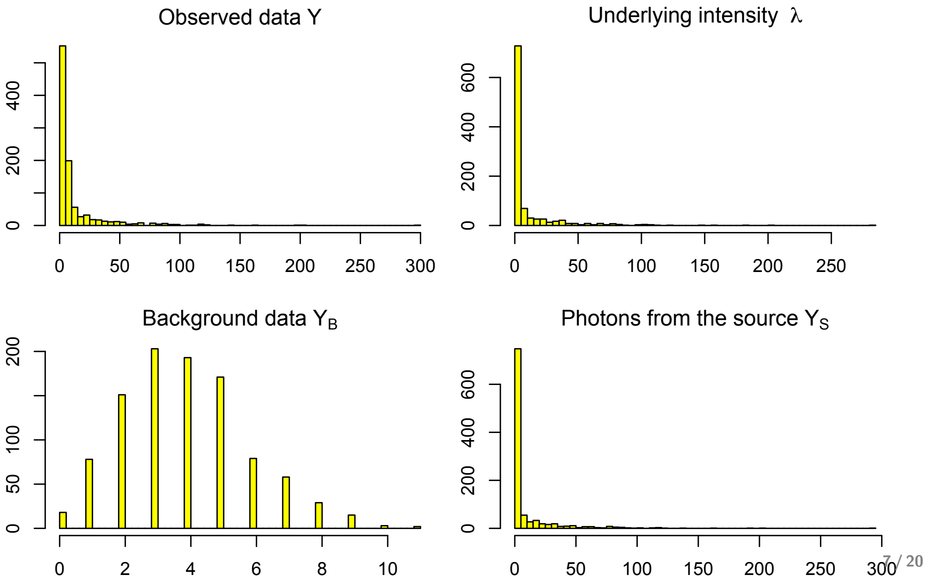
Frequency Coverage: Simulation Setting 1



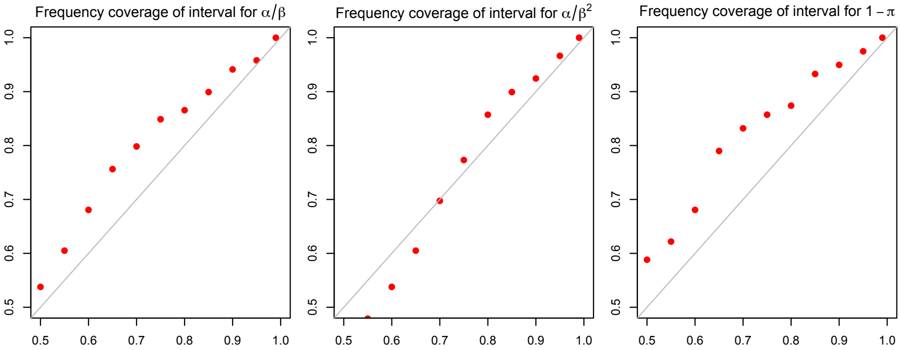
Frequency Coverage: Simulation Setting 1



Frequency Coverage: Simulation Setting 2



Frequency Coverage: Simulation Setting 2



Identifying the Existence of Dark Sources

- Hypothesis Testing:

$$H_0 : 1 - \pi = 0, \quad H_a : 1 - \pi > 0.$$

- H_0 corresponds to M_0 with the second level

$$\lambda_i | \alpha, \beta \sim Gamma(\alpha, \beta)$$

- H_a corresponds to M_a with the second level

$$\lambda_i | \alpha, \beta, \pi \begin{cases} = 0, & \text{with probability } 1 - \pi; \\ \sim Gamma(\alpha, \beta), & \text{with probability } \pi. \end{cases}$$

Hypothesis Testing

- Likelihood Ratio Test Statistics:

$$R = \frac{L_a(\hat{\alpha}_{MLE}, \hat{\beta}_{MLE}, \hat{\pi}_{MLE}|Y)}{L_0(\hat{\alpha}_{MLE}, \hat{\beta}_{MLE}|Y)}$$

- What's the distribution of R or $log(R)$ under H_0?
- p-value is used to measure how likely we are to see a value of the test statistics as extreme as the observed value under H_0.

$$\text{p-value} = P(R \geqslant R^{obs}|H_0)$$

The Distribution of R under H_0

- Simulate N data sets Y^{rep} under H_0 and compute R^{rep} for each of the N data sets.
- P-value can be approximated by

$$\text{p-value} \approx \frac{\#\{i : R_i^{rep} \geqslant R^{obs}\}}{N}$$

- However, we can not simulate data sets under H_0 because α and β are unknown.
- Instead, we simulate $Y^{rep} \sim M_0$ with $\alpha, \beta \sim P_0(\alpha, \beta | Y^{obs})$. So the resulted "p-value" is the posterior predictive p-value under the M_0.

Calculation of R: Maximum likelihood under M_0

- Likelihood under M_0:

$$L_0(\alpha, \beta | Y^{rep}) = \int P(Y^{rep}, \lambda | \alpha, \beta) d\lambda$$

$$\propto \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^{n} \int e^{-(b_i+\beta)\lambda_i} \frac{(a_i\xi + b_i\lambda_i)^{Y_i^{rep}}}{Y_i^{rep}!} \lambda_i^{\alpha-1} d\lambda_i$$

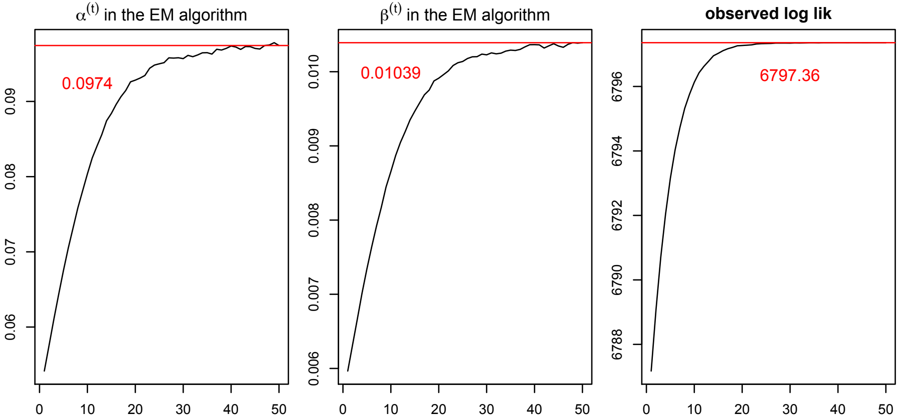
- *EM* algorithm (λ's are treated as missing data).
- In the E-step, we need to find

$$T_1^{(t)} = E_t(\sum_{i=1}^{n} \lambda_i | Y^{rep}) \text{ and } T_2^{(t)} = E_t(\sum_{i=1}^{n} log(\lambda_i) | Y^{rep})$$

- Simulation to estimate $T_1^{(t)}$ and $T_2^{(t)}$:

$$\text{Gibbs sampling: } \lambda_i^{(t)} \sim P(\lambda_i | \alpha^{(t)}, \beta^{(t)}, Y^{rep}), i = 1, \cdots, n$$

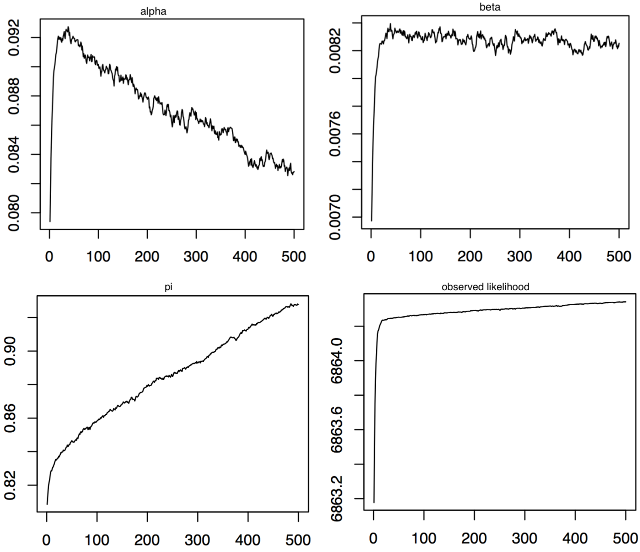
Calculation of R: Maximum likelihood under M_0



Calculation of R: Maximum likelihood under M_a

- *EM* algorithm (λ's are treated as missing data)
- Gibbs sampling: $\lambda_i^{(t)} \sim P(\lambda_i | \alpha^{(t)}, \beta^{(t)}, \pi^{(t)}, Y^{rep})$
- However,
  - Each step in the EM algorithm is very slow
  - EM algorithm converges very slowly

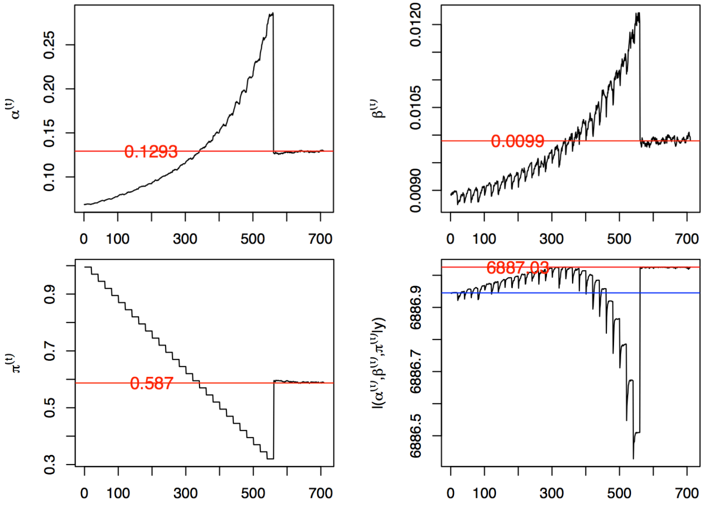
Calculation of R: Maximum likelihood under M_a



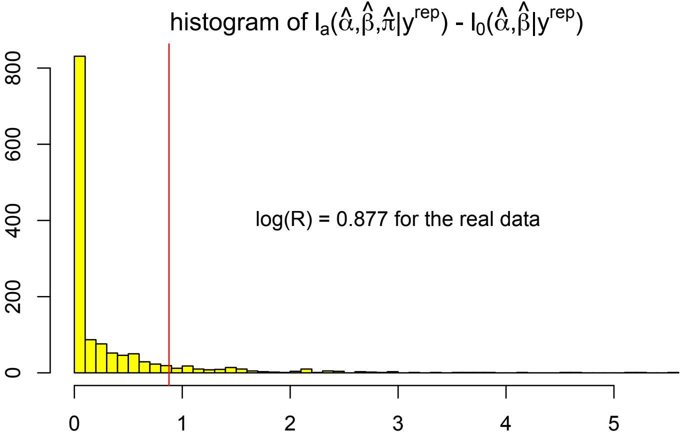
A More Efficient Method to Calculate the Maximum likelihood under M_a

- Observation: for a fixed π, the *EM* converges very fast.
- A more efficient algorithm:
  1. Explore the space of π: fix π at a range of values $\pi_1, \pi_2, \cdots, \pi_K$ and compute the
  $$L_k = L_a(\hat{\alpha}_k, \hat{\beta}_k, \pi_k | Y^{rep})$$
  2. Choose k^* such that
  $$k^* = arg \max_k L_a(\hat{\alpha}_k, \hat{\beta}_k, \pi_k | Y^{rep})$$
  3. Doing the complete EM algorithm with starting points
  $$\pi^{(0)} = \pi_{k^*}, \alpha^{(0)} = \hat{\alpha}_{k^*}, \beta^{(0)} = \hat{\beta}_{k^*}$$

A More Efficient Method to Calculate the Maximum likelihood under M_a

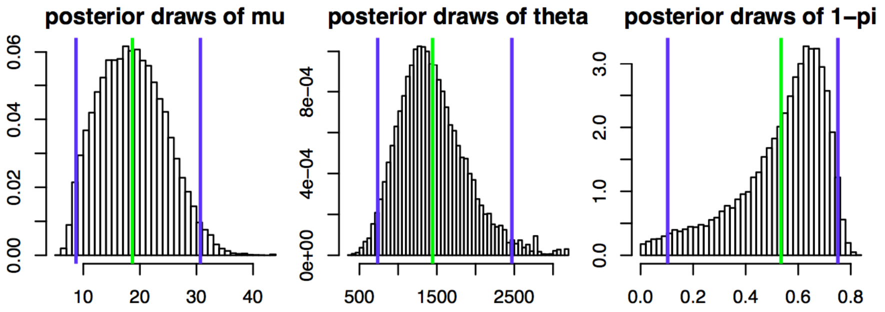


Posterior Predictive P-value



posterior predictive p-value $= P(log(R^{rep}) \geqslant log(R^{obs})) = 0.105$

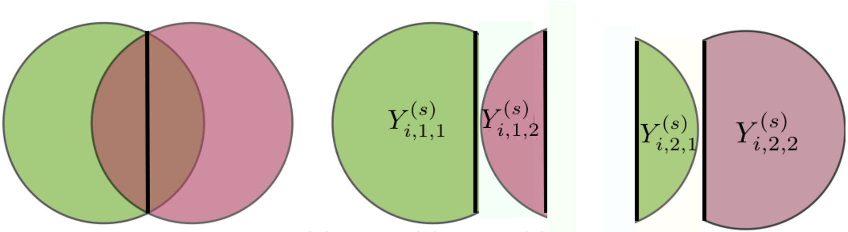
MAP $\approx$ MLE



$$\frac{\hat{\alpha}_{MLE}}{\hat{\beta}_{MLE}} = 20.22, \quad \frac{\hat{\alpha}_{MLE}}{\hat{\beta}^2_{MLE}} = 1451.2, \quad 1 - \hat{\pi}_{MLE} = 0.624.$$

EM Posterior under the null

Model for dealing with Overlapping Sources



$$Y_{i,j} = Y^{(s)}_{i,j,1} + Y^{(s)}_{i,j,2} + Y^{(b)}_{i,j}$$

$$Y^{(s)}_{i,j,k} \sim Pois(b_{i,j,k}\lambda_{i,k}),$$

where $b_{i,j,k} = b_{i,k}c_{i,j,k}$, $b_{i,k}$ is the effective area and $c_{i,j,k}$ is the expected proportion of photons from source k counted in $Y_{i,j}$

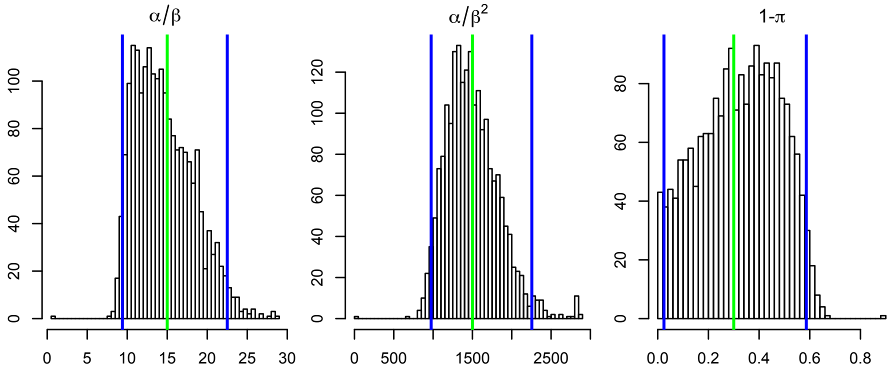
Model for dealing with Overlapping Sources

Level I Model:

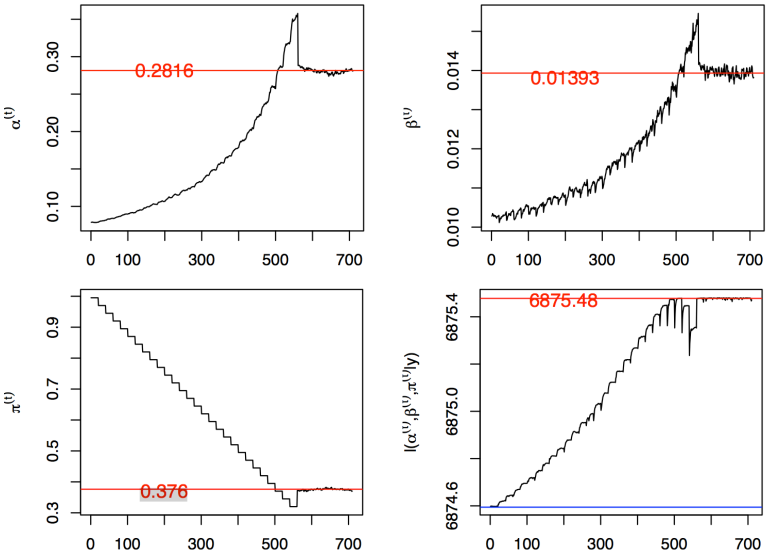
$$
\begin{aligned}
& Y_{i,j} = Y_{i,j}^{(s)} + Y_{i,j}^{(b)},\ \ i = 1, \cdots, n,\ \ j = 1, \cdots, n_i, \\
& Y_{i,j}^{(b)} | \xi \sim Pois(a_i \xi), \\
& Y_{i,j}^{(s)} = \sum_{k=1}^{n_i} Y_{i,j,k}^{(s)} \\
& Y_{i,j,k}^{(s)} | \lambda_{i,k} \sim Pois(b_{i,j,k} \lambda_{i,k}),\ \ k = 1, \cdots, n_i,
\end{aligned}
$$

Simulation Results

35% n_i's are 1, 5% n_i's are 2 and 65% n_i's are 3.



Maximum Likelihood under M_a for the Real Data



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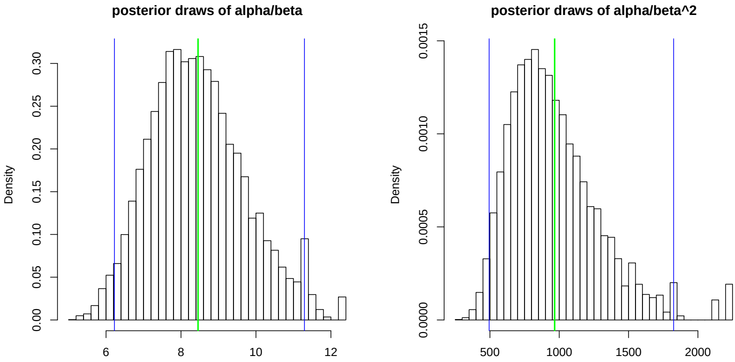
Posterior Distribution under M_0



$$\frac{\hat{\alpha}_{MLE}}{\hat{\beta}_{MLE}} = 7.87, \quad \frac{\hat{\alpha}_{MLE}}{\hat{\beta}^2_{MLE}} = 787,$$

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