Fun Statistics with Fourier Spectra

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Almost all things in the universe are variable

Bolin et al (2017)

Angus et al (2018)


Vaughan et al (2016)

Strohmayer & Watts (2005)

Almost all things in the universe are variable
jet physics and particle acceleration

stellar winds

radiative processes

general relativity

relativistic plasmas + magnetic fields
X-ray Spectra Vary with Time

Malzac (2008)
X-ray Spectra Vary with Time

Malzac (2008)
X-ray Spectra Vary with Time

Malzac (2008)
X-ray Spectra Vary with Time

![Graph showing x-ray spectra varying with photon energy and time, highlighting "soft state" and "hard state" phases.](image)

Malzac (2008)
X-ray Spectra Vary with Time

Malzac (2008)
X-ray Spectra Vary with Time

Malzac (2008)

Photor Energy [keV]

Photon Energy * Flux [detector units]

“hard state”

“soft state”

Malzac (2008)

high-frequency/low-frequency X-ray brightness
X-ray Spectra Vary with Time

Malzac (2008)
Fourier Analysis tells us about variability!*
Fourier Analysis tells us about variability!* 
*for evenly sampled time series
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*for evenly sampled time series

\[ x(t) = \frac{1}{N} \sum_j a_j \cos(\omega_j t - \phi_j) = \frac{1}{N} \sum_j (A_j \cos \omega_j t + B_j \sin \omega_j t). \]
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a_j = \sum_{k=0}^{N-1} x_k e^{2\pi ijk/N}
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A_j = \sum_{k} x_k \cos \omega_j t_k
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B_j = \sum_{k} x_k \sin \omega_j t_k,
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*for evenly sampled time series 

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\[ B_j = \sum_k x_k \sin \omega_j t_k, \]

\[ a = \sqrt{A^2 + B^2} \]
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\[ a_j = \sum_{k=0}^{N-1} x_k e^{2\pi i j k / N} \]

periodogram:

\[ P_j \equiv \frac{2}{N_{ph}} |a_j|^2 \quad j = 0, \ldots, \frac{N}{2} , \]
Fourier Analysis tells us about variability!*

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statistical distribution?
Fourier Analysis tells us about variability!* 

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assume many data points

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P_j \equiv \frac{2}{N_p h} |a_j|^2 \quad j = 0, \ldots, \frac{N}{2},
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Fourier Analysis tells us about variability!*  

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~ Gaussian
Fourier Analysis tells us about variability!* 

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\[ P_j = \frac{2}{N_{ph}} |a_j|^2 \quad j = 0, \ldots, \frac{N}{2}, \]

\[ |a_j|^2 = A^2 + B^2 \]

~ Gaussian
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\[ \chi^2 \text{ with 2 degrees of freedom} \]
Fourier Analysis tells us about variability!*

*for evenly sampled time series

white noise
Fourier Analysis tells us about variability!*

*for evenly sampled time series

---

**Periodic signal light curve**

- Time [s] vs. Count rate [counts/s]
- Values range from 0 to 250

**Periodogram with Periodic Signal**

- Frequency [Hz] vs. Leahy-normalized Power
- Peaks at specific frequencies indicate periodicity

---

**periodic signal**
Fourier Analysis tells us about variability!* 

*for evenly sampled time series

periodic signal
Fourier Analysis tells us about variability!*  

*for evenly sampled time series

**Red noise light curve**

**Red Noise Periodogram**

**Count rate [counts/s]**

**Leahy-normalized Power**

**Time [s]**

**Frequency [Hz]**
Fourier Analysis tells us about variability!*

*for evenly sampled time series

quasi-periodic oscillations
Figure 2. 

a) Graphic illustrating where various states appear within the power-colour diagram. The area of overlap between the hardest and softest states is also indicated.

b) Power colour-colour plot for all observations of the transient objects within the sample with labels indicating 20-degree azimuthal or 'hue' regions from which the power spectra given in c were found. The plot is colour-coded for each 20° bin with the same colours used in c.

c) Example power spectra for each of the 20 degree ranges of hue around the power-colour diagram. Colours and indices refer to the 20° angular bins used in b. Further examples are given in the Appendix.

Figure 1 shows the power colour values for two particular frequency ratios, Power colour ratio 1 (PC1) is defined as variance in 0.25-2.0 Hz/0.0039-0.031 Hz and ratio 2 (PC2) is variance in 0.031-0.25 Hz/2.0-16.0 Hz. These particular ratios not only compare all four broad frequency bands used in the initial analysis, making the most of the available data, but are also separated in frequency. This plot is colour coded according to object, and the similarity in power-spectral evolution throughout bursts be-
So, we’re done, right?
Hercules X-1

NuSTAR combined light curve

Count rate [counts/s]

Time in days since MET 85740108
Hercules X-1

Huppenkothen & Bachetti, in press
Hercules X-1

NuSTAR combined light curve

Count rate [counts/s]

Time in days since MET 85740108

Averaged PSD (3260 segments)

Leahy Power

Frequency [Hz]
After detection of a photon, the detector is "dead" for a short interval

Chaplin et al (2012)
After detection of a photon, the detector is “dead” for a short interval.
Dead time affects mean and variance of the periodogram
Can we fix this?
Yes (ish)
Having Two Detectors Helps!
Signal is the same, but the measurement noise is different!
Compute cross spectrum instead of periodogram

\[ F_x(j)F^*_y(j) = \frac{1}{2} \left\{ (A_{xj}A_{yj} + B_{xj}B_{yj}) + i(A_{xj}B_{yj} - A_{yj}B_{xj}) \right\} \]
Compute cross spectrum instead of periodogram

\[ \mathcal{F}_x(j)\mathcal{F}^*_y(j) = \frac{1}{2} \left\{ (A_{xj}A_{yj} + B_{xj}B_{yj}) + i(A_{xj}B_{yj} - A_{yj}B_{xj}) \right\} \]

phase/time lag
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phase/time lag

\[ C_j = \frac{1}{2} (A_{xj}A_{yj} + B_{xj}B_{yj}) \]

cospectrum
What is the statistical distribution of the cospectrum?
periodogram
periodogram

$|a_j|^2 = A^2 + B^2$
periodogram

\[ |a_j|^2 = A^2 + B^2 \]

\( \chi^2 \) distributed
periodogram

\[ |a_j|^2 = A^2 + B^2 \]

\[ \chi^2 \text{distributed} \]

cospectrum

\[ C_j = \frac{1}{2} (A_{xj} A_{yj} + B_{xj} B_{yj}). \]
periodogram

$$|a_j|^2 = A^2 + B^2 \quad \chi^2 \text{ distributed}$$

cospectrum

$$C_j = \frac{1}{2} (A_{xj} A_{yj} + B_{xj} B_{yj})$$

$$A_{xj}^2 \neq A_{xj} A_{yj}$$
**periodogram**

\[ |a_j|^2 = A^2 + B^2 \]

\[ \chi^2 \text{distributed} \]

**cospectrum**

\[ C_j = \frac{1}{2} (A_{xj} A_{yj} + B_{xj} B_{yj}) \]

\[ A_{xj}^2 \neq A_{xj} A_{yj} \]

\[ \text{not } \chi^2 \text{distributed} \]
we know\[A_{xj}, B_{xj} \sim \mathcal{N}(0, \sigma_x^2)\]
we know

\[ A_{xj}, B_{xj} \sim \mathcal{N}(0, \sigma_x^2) \]

set

\[ Z = A_{xj}A_{yj}, \quad Q = B_{xj}B_{yj} \]
we know that the PDF is:

\[ A_{xj}, B_{xj} \sim \mathcal{N}(0, \sigma_x^2) \]

set

\[ Z = A_{xj}A_{yj}, \quad Q = B_{xj}B_{yj} \]

then the PDF is:

\[ P_Z(z) = \frac{K_0 \left( \frac{|z|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y}, \]

Watson (1922); Wishart & Bartlett (1932)
we know

\[ A_{xj}, B_{xj} \sim \mathcal{N}(0, \sigma_x^2) \]

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Bessel function

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then the PDF is:

\[ P_Z(z) = \frac{K_0 \left( \frac{|z|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y}, \]

the distribution for

\[ Z + Q \]
we know

\[ A_{xj}, B_{xj} \sim \mathcal{N}(0, \sigma^2_x) \]

set

\[ Z = A_{xj}A_{yj}, \quad Q = B_{xj}B_{yj} \]

then the PDF is:

\[
P_Z(z) = \frac{K_0 \left( \frac{|z|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y},
\]

the distribution for \( Z + Q \) is the convolution of PDFs:

\[
p_{Z+Q}(z) = p_Z \ast p_Q(z)
\]

Bessel function

Watson (1922); Wishart & Bartlett (1932)
\[ p_{z+Q}(z) = p_z \ast p_Q(z) \]
combine

\[ p_{Z+Q}(z) = p_Z \ast p_Q(z) \]

and

\[ P_Z(z) = \frac{K_0 \left( \frac{|z|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y}, \]
\[ p_{Z+Q}(z) = p_Z \ast p_Q(z) \]

and

\[ P_Z(z) = \frac{K_0 \left( \frac{|z|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y}, \]

to get

\[ p_{Z+Q}(z) = \int_{-\infty}^{+\infty} \frac{K_0 \left( \frac{|l|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y} \frac{K_0 \left( \frac{|z-l|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y} \, dt \]
$p_{Z+Q}(z) = p_Z * p_Q(z)$

and

$P_Z(z) = \frac{K_0 \left( \frac{|z|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y}$

to get

$p_{Z+Q}(z) = \int_{-\infty}^{+\infty} \frac{K_0 \left( \frac{|t|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y} \frac{K_0 \left( \frac{|z-t|}{\sigma_x \sigma_y} \right)}{\pi \sigma_x \sigma_y} dt$
“The sum of two random variables is equivalent to the multiplication of its moment-generating functions.

— no astronomer, ever
moment-generating function:

\[ M_Z(t) := \mathbb{E}[e^{tZ}] \]
moment-generating function:

\[ M_Z(t) := \mathbb{E}[e^{tZ}] \]

multiply MGFs of two variables:

\[ M_S(t) = M_Z(t)M_Q(t) \]
moment-generating function:

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multiply MGFs of two variables:

\[ M_S(t) = M_Z(t)M_Q(t). \]

for the cospectrum:

\[ M_C(t) = M_Z(t)M_Q(t) = \frac{1}{1 - t^2\sigma_x^2\sigma_y^2}. \]

Seijas-Macías & Oliveira (2012)
moment-generating function:

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for the cospectrum:

\[ M_C(t) = M_Z(t)M_Q(t) = \frac{1}{1 - t^2\sigma_x^2\sigma_y^2}. \]

This is a Laplace distribution!

\[ p(C_j|0, \sigma_x\sigma_y) = \frac{1}{\sigma_x\sigma_y} \exp \left( -\frac{|C_j|}{\sigma_x\sigma_y} \right) \]

Seijas-Macías & Oliveira (2012)
... why are we doing this again?
χ² distribution
χ² distribution

Laplace distribution
significance threshold matters!
what about averaged cospectra?
\[ f_{X_n}(x) = \frac{ne^{-|nx|}}{(n-1)!2^n} \sum_{j=0}^{n-1} \frac{(n-1+j)!}{(n-1-j)!j!} \frac{|nx|^{n-1-j}}{2^j}, \quad x \in \mathbb{R} \]
Gaussian approximation works for large $N$
so now we’re done, right?
so now we’re done, right?

... not quite!
1) Equations so far only work for white noise.

2) The cospectrum only fixes the mean in the dead time case, not the variance!
What about stochastic variability?
What about stochastic variability?

\[ MC_j(t) = \frac{1}{[1 - (1 + r)t][1 + (1 - r)t]} \cdot \]
What about stochastic variability?

\[ M_{C_j}(t) = \frac{1}{[1 - (1 + r)t][1 + (1 - r)t]} \cdot \]

correlation coefficient between Fourier amplitudes
What about stochastic variability?

\[ M_{C_j}(t) = \frac{1}{[1 - (1 + r)t][1 + (1 - r)t]} \cdot \]

correlation coefficient between Fourier amplitudes

\( r = 0 \) for white noise
\( r = 1 \) for power spectra
statistical distribution depends on $r$, which depends on power spectral shape!
statistical distribution depends on \( r \), which depends on power spectral shape!

also, there is no existing closed-form solution for the PDF
statistical distribution depends on $r$, which depends on power spectral shape!

also, there is no existing closed-form solution for the PDF
Can we model the Fourier amplitudes directly?
Can we model the Fourier amplitudes directly?

\[ p(\{A_{xj}, A_{yj}, B_{xj}, B_{yj}\}_{j=0}^{N/2} | \theta, \lambda_{\text{phot}}) = \sum_{j=0}^{N/2} \left[ -\log(2\pi|C|) - ([A_{xj}, A_{yj}]^TC^{-1}[A_{xj}, A_{yj}]) - ([B_{xj}, B_{yj}]^TC^{-1}[B_{xj}, B_{yj}]) \right] \]

\[
C = \begin{bmatrix}
\sigma_{sj} + \sigma_n & \sigma_{sj} \\
\sigma_{sj} & \sigma_{sj} + \sigma_n
\end{bmatrix}
\]
Can we model the Fourier amplitudes directly?

\[
p(\{A_{xj}, A_{yj}, B_{xj}, B_{yj}\}_{j=0}^{N/2} | \theta, \lambda_{\text{phot}}) = \sum_{j=0}^{N/2} \left[- \log(2\pi |C|) - ([A_{xj}, A_{yj}]^T C^{-1} [A_{xj}, A_{yj}]) - ([B_{xj}, B_{yj}]^T C^{-1} [B_{xj}, B_{yj}])\right]
\]

\[
C = \begin{bmatrix}
\sigma_x + \sigma_n & \sigma_x \\
\sigma_x & \sigma_x + \sigma_n
\end{bmatrix}
\]

depends on \(P(v)\)
But: we can correct the periodogram (and the cospectrum) in some cases!
Fourier Amplitude Difference Correction

red noise, no dead time

Bachetti & Huppenkothen (2018)
Fourier Amplitude Difference Correction

red noise, no dead time

Bachetti & Huppenkothen (2018)
Fourier Amplitude Difference Correction

red noise, no dead time

Bachetti & Huppenkothen (2018)
Fourier Amplitude Difference Correction

red noise, no dead time  
white noise, dead time

Bachetti & Huppenkothen (2018)
Fourier Amplitude Difference Correction

red noise, no dead time

white noise, dead time

Bachetti & Huppenkothen (2018)
Fourier Amplitude Difference Correction

red noise, no dead time

white noise, dead time

Bachetti & Huppenkothen (2018)
Fourier Amplitude Difference Correction

red noise, no dead time

white noise, dead time

use to correct periodogram

Bachetti & Huppenkothen (2018)
**Fourier Amplitude Difference Correction**

Bachetti & Huppenkothen (2018)
Caveat: this overestimates the rms amplitude when both flux and rms are very large
Conclusions

- statistics with Fourier spectra is fun!
- use the cospectrum to do timing of bright sources in the presence of dead time when more than one detector is available (Bachetti+, 2015)
- the (averaged) cospectrum requires different statistical distributions for significance testing (Huppenkothen+Bachetti, arXiv:1709.09666)
- there is currently no closed-form solution for red noise cospectra (future work)
- but red noise periodograms can be corrected using the FAD technique (Bachetti+Huppenkothen, arXiv:1709.09700)
## Python API for XSPEC
Develop a modular python API to use XSPEC (a popular X-ray astronomy tool) in python workflows

## Optimize Stingray for Large Datasets
Optimize tools in the Stingray library for use on large datasets from new X-ray space missions

## Phase-resolved oscillations
Implement method to calculate the phase of oscillatory phenomena with non-constant frequency, and calculate phase-resolved spectra

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<th>dhuppenkothen</th>
<th>matteobachetti</th>
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